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ACTA MATHEMATICA 13

MATHEMATICAL OPPORTUNITIES – MATHEMATICAL AWARENESS A QUESTION OF PRACTICE

JOHN ANDERSEN

ABSTRACT. Sometimes you hear the phrase "Mathematics is everywhere". It could for instance be stated by a committed teacher eager to promote the subject to students. Unfortunately not all students immediately realize the existence of mathematics in their surroundings. The mathematics in a context is not solely determined by the particular context. The things around do not automatically impose mathematics on a passive mind. You have to practice a lot to develop habits promoting awareness concerning mathematical opportunities. This article deals with examples in connection with description of motion by the means of Cartesian coordinate systems and parametric equations.

Introduction

In the report from the Comenius project Math2Earth funded by the European Commission ([1], [2]) you will find a lot of very different examples showing a wide range of opportunities for experiencing mathematics in varying contexts. The collection of examples is written with the intention to inspire teachers working at different levels in the educational system.

Learning to identify dressed up mathematics in different contexts is not an easy task. It is not something that happens automatically as an outcome of solving standard problems from mathematical textbooks and proving theorems in pure mathematics. You need to work through a lot of concrete examples. [3].

At not too advanced levels it seems that some subjects are more easily recognised than others outside classroom context. Looking at the watchmakers shop window (Fig. 1) topics such as money (prices of watches) or time (natural thing to think about when you see a watch) geometrical shapes (the decorative stars) more or less immediately comes to mind.

Figure 1: Photograph from a watchmakers shop window in Vienna

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In [3] you will find some general considerations concerning mathematical awareness and related problems. In this article some specific examples on activities designed to study mathematical descriptions of motion and localisation are discussed. Part of it is a further development of ideas presented in [4].

Parametric equations and motion in teacher education

The point of departure for the examples presented is activities from teacher education at a class consisting of 15 students at their second year of studies at the Bachelor Program of Teacher Education at VIA University Collage in Aarhus, Denmark. The educated teachers will normally be going to teach in the Danish primary and lower secondary school.

We were about to study the following point from the curriculum: *d) Geometry, including analytical geometry, parametric equations and trigonometry.*

Usually the study material is taken from textbooks (theory, examples and problems) and problems from earlier examinations. To promote motivation the students had asked if we could work with a problem from the recent written examination.

As you will see a lot of information technology is involved in the classroom work, such as spreadsheets, geometry software, digital cameras and GPS devises. The technology makes it possible to freeze the fleeting moments of the motions considered and to present results.

The Task

The title of the task is "Two Ships" and the mathematical subject that was treated was the description of plane movements by means of parametric equations and Cartesian coordinate system. Since one of the aims of teacher education is to enhance students' ability to identify mathematics in different contexts the mathematical content of the particular problem was dressed up in the context of problems about danger of collision of ships.

To introduce the context there was a CD with some information from a web page about safe sailing. If you want to see for yourself you can follow the link [5]. At the examination the relevant part of the website was included on a CD since the students were not allowed access to the web during examination.

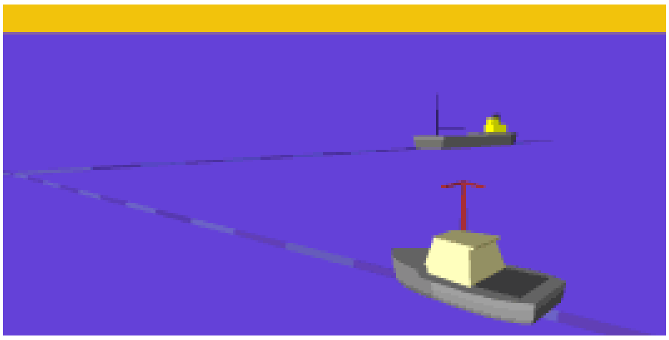


Figure 2 from a animated video showing a situation where two ship are on colliding courses [3]

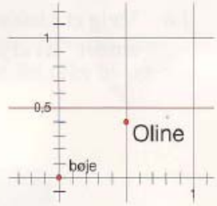
A part of the original problem[1] is shown below in Figure 3.

Koordinatsystemet på figur 2.2 er placeret, så en bøje er sat i (0,0).

Et skib med navnet Oline har til tiden $t = 0$ positionen 0,5 sømil øst for bøjen og 0,4 sømil nord for.

Oline sejler med en fart på 8 sømil i timen med kurs 55°.

På figur 2.2 er Oline og bøjen placeret til tiden $t = 0$.



Figur 2.2

2.1 Indtegn i et geometriprogram (eller på svararket) Olines position for hvert 10. minut for t i intervallet [0;1].

2.2 Vis, at Olines bevægelse kan beskrives ved parameterfremstillingen

$$(x(t), y(t)) = (6{,}55t + 0{,}5 \ ; \ 4{,}59t + 0{,}4)$$

Figure 3: A part of the original formulation of the examination problem – see translation below

The coordinate system at figure 2.2 is positioned in such a way that a buoy is located in (0,0).
A ship named Oline occupies at time t = 0 a position 0.5 nautical miles east to the buoy and 0.4 nautical miles to the north.
Oline is saling along with a speed at 8 nautical miles per hour.
On figure 2.2 Oline and the buoy are shown at time t = 0.

2.1 Draw by means of geometry software the Position of Oline for every 10th minute for t in the interval [0; 1].
2.2 Show that motion of Oline can be described by the parametric equations

$$(x(t),y(t)) = (6.55t + 0.5 \ , \ 4.59t + 0.4)$$

Figure 4: Translation of the text in Figure 3.

The next part of the task was to decide whether or not Oline was about to collide with another ship Petra with trajectory given by another set of parametric equations. The details are omitted for brevity.

What is in the curriculum is not ships and navigation but parametric equations which meant that the students not were familiar with ships and navigation but with solving tasks involving parametric equations in some standard form. It appeared that the dressing up in the nautical context did confuse most of the students. It was not because they couldn't manage the "pure" mathematical content when the problem was stripped of the context. In a traditional setup the problem could look something like (Fig. 5). It is simplified a bit compared to the original problem to save space.

[1] The problem is from an examinaton, May 2009. The class worked with the problem in Oktober 2009. Due to copyright reasons the problem is not available in electronic form so no link can be given and it is to lengthy to reproduce completely in this article.

A line passes through the point A(0.4 , 0.5) and makes an angle v = 55   with the positive x-axis.

a) Show that (x(t),y(t)) = (6.55t + 0.5 , 4.59t + 0.4) determines parametric equations for the line.

Another line is given by the parametric equations

$$\left(x(t), y(t)\right) = \left(-\sqrt{5} \cdot t + 5.5, 2\sqrt{5} \cdot t + 1.0\right)$$

b) Draw the line in the same coordinate system as the line from a).

c) Determine the point of intersection between the two lines.

Figure 5: A traditional 'context free' formulation of the 'Two Ships' problem.

The fact that confusion arises in the minds of students' by introducing some kind of real world context for the mathematical content indicate that one should avoid these attempts of contextualising. Or one could train the students so that whenever they encounter these wrapped up problems they immediately strips them of strange context formulations and identify the bare calculating task. A kind of strategy that could be named: 'I must get rid of the context'.

Since the main reasons for teaching mathematics is that students have to develop mathematical skills to be used in other contexts than mathematical lessons this would be a poor strategy. Another mode of attack is to introduce still more contexts training students ability to recognise the same mathematical concepts and methods in different set ups.

Introducing other contexts for description of motion by parametric equations

Going in and out of different contexts can cause a lot of trouble. The students were only half way through the course in mathematics so there were room for further training in dealing with parametric equations.

As another approach we looked for a situation where the actual moving bodies were more manageably. Real ships were not at hand in classroom. Time and other resources did not allow for going out in the world to find some real ships at the moment.

Instead we experimented with rubber balls to play the role of ships. Two students roll a red and a green ball over the floor in directions that indicate a risk of collision. A third student films the rolling balls with a digital camera. By means of Windows Movie Maker the individual frames from the film is picked out. Six of them are shown below (Fig. 6).

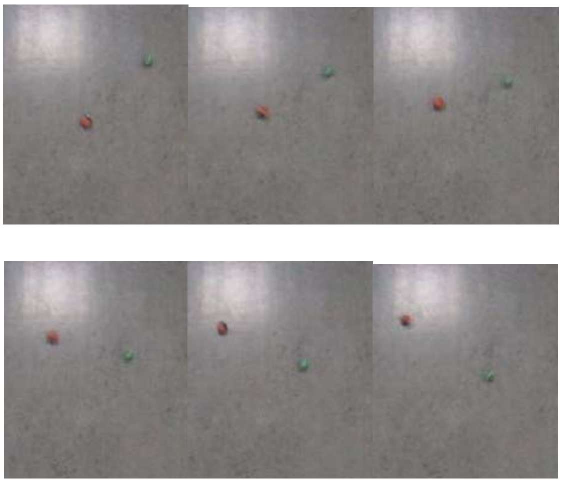


Figure 6: Six frames from a video recording of two rolling rubber balls

In the individual frames the motion is frozen at equally spaced moments. Next step is to transfer all the frozen moments into one frame. This is done by importing the pictures one by one to some geometry software marking of the individual positions in a coordinate system. In the actual instance was used Geometers Sketchpad:

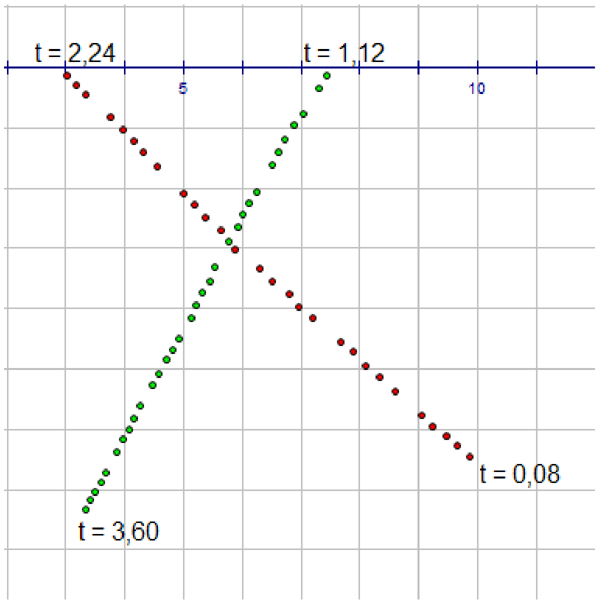


Figure 7: Positions of the two rubber balls frame by frame transferred to a coordinate system.

First you will think that it must be a very lengthy process to carry through, but believe it or not, after the first fumbling moments you develop routines that makes the work run smooth and at a reasonable time. See also [4].

Now you are in a position resembling the position emerging when you fill in points from a parametric equation into a coordinate frame. The main difference is that the points in this ball game case come from the actual movement initiated by the students themselves. The idea by going this direction is to enhance the connection between the two positions.

One notices that the points are not evenly spaced (there is a gap for each five frames) on the two traces as one would expect since the timeline in Windows Moviemaker marked evenly spaced time moments. Especially for the green ball (lowest left corner) one can see that the ball is slowing down. We didn't go further into this but calculated on assuming uniform movement not to lose focus on the connection to the problem of linear motion from the exam problem. But if time and interest had suggested further investigation of these phenomenons it would have been possible.

Next step was to calculate parametric equations for the two rolling balls. Unfortunately the time at disposal did not allow us to do this in classroom so I, the teacher, did it at my office (but given more time I would have made it a task for the students). I shall not go into details here. Just showing a screen shot of one of the parametric equations which were set up by means of the calculator inherent in the geometry software. See Fig. 8.

$$\left(x_{t=2,24}-x_{t=0,08}\right)\cdot\frac{(t-0,08)}{2,24-0,08}+x_{t=0,08}=3,14$$

$$\left(y_{t=2,24}-y_{t=0,08}\right)\cdot\frac{(t-0,08)}{2,24-0,08}+y_{t=0,08}=-0,25$$

Figure 8: Parametric equations for the movement of one of the rubber balls.

It then was made possible to animate the movement described by the parametric equation in real time and I could show the students both the real film of the rolling balls and the animation made by parametric equations.

Introducing GPS[2] technology

Another activity designed to bring the mathematical technique of plotting pairs of numbers as points in a coordinate system in contact with situations from students everyday can be created by introducing GPS technology.

Figure 9: The GPS devise used (GARMIN etrex Vista HCx)

[2] GPS is an abbreviation for Global Positioning System

A hand held GPS device can record the time and position data at its location at user set intervals.

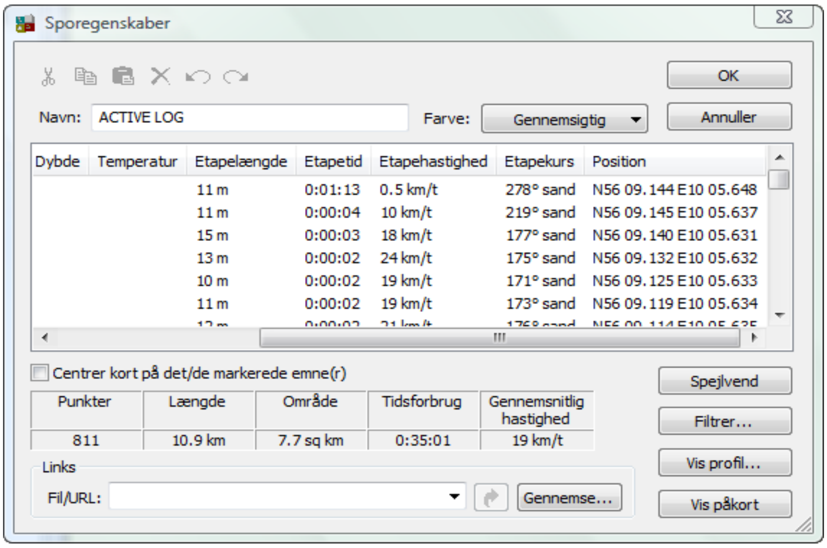


Figure 10: Data from GPS devise (for each time interval is recorded travelled distance, length of time interval, speed, compass direction, geographical position and local time (not shown)).

By means of software coming with the GPS device it is possible to transfer the data to Google Earth and see the trajectory of a bike ride on a map. See Fig. 11.

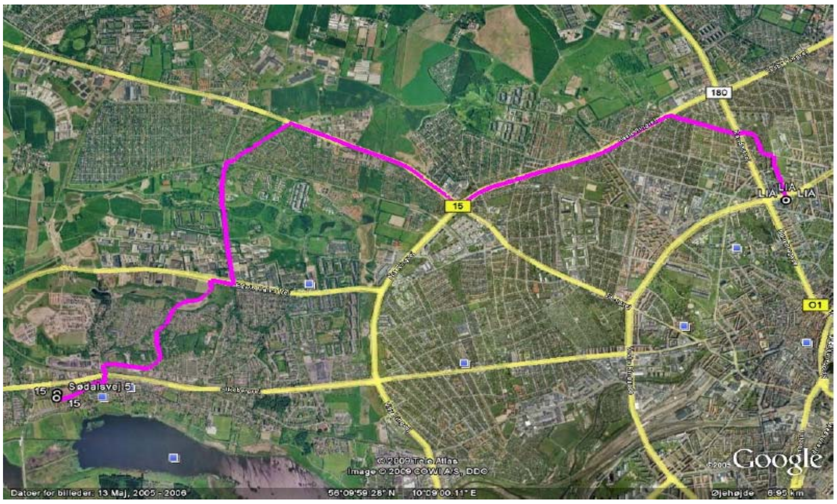


Figure11: Bike trip from my home to VIA. Trajectory starting left at N 56 9.144' E 10 5.654'.

Most of the mathematics is build into the device and the software, but it is possible, even on a rather elementary level to show the connection between plotting of coordinate sets and the trajectory. For instance position coordinates can be imported in a spreadsheet and the trajectory can be graphed by means of the spreadsheet tools and compared to the Google Earth path.

	A	B	C	D	E
1	Point no.	Time from start (sec)	x-coord. (m)	y-coord.(m)	Elevation over sealevel (m)
2	1	0	0	0	13
3	2	73	-11	2	12
4	3	77	-17	-7	12
5	4	80	-16	-22	12
6	5	82	-15	-35	12
7	6	84	-13	-45	11
8	7	86	-12	-56	11
9	8	88	-11	-68	11

Figure 12: GPS data transferred to spreadsheet

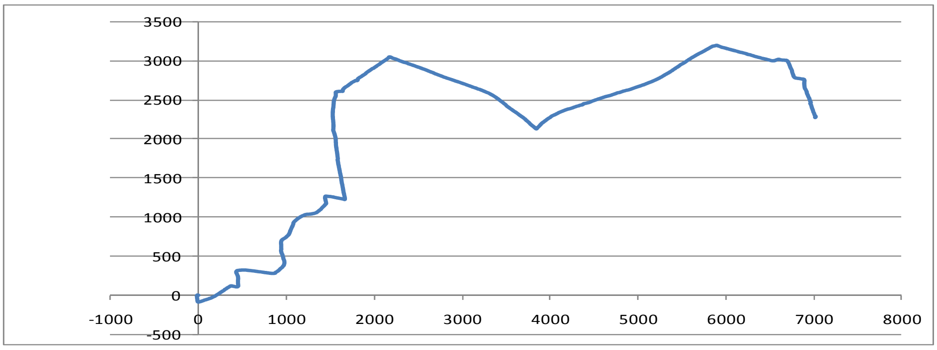


Figure 13: Trajectory for bike ride graphed in Excel 2007

This is not done with the purpose to expel Google Earth and 'do it yourself ' by Excel but to show that coordinate systems are involved in GPS techniques. Activities like this may enhance students' awareness of the usability of coordinate systems outside classroom. It is possible to describe trajectories of travels by more low technological means such as stopwatches and maps. Sometimes his can make things more clear although it may seem more cumbersome to do it manually. See [4].

From the GPS data you can go into more detailed study of your journey. For instance you can produce and study graphs over time versus distance, time versus speed, distance versus speed, distance versus elevation, ascent/descent versus speed. Or you can compare graphs from different ways of travelling the same route (walk, bike, car, bus).

Another direction you may take is examining the problem of how at all it is possible to measure your geographical position with a devise like the one on Fig. 9. Pretty soon you will be involved in rather complicated applications of analytical geometry etc. Of course the focus and choice of subject should be in accordance with students' level and abilities and with curricula so the suggested activities are not meant to be presented to all students. It can be part of the teacher's pool of possibilities for picking suitable activities with the purpose of engaging and inspiring students.

Reversing the GPS activity

Another way round would be to start with a sufficiently simple parametric equations and ask the students to perform the motion described. For a linear motion it became the activity 'Run a Graph' tested in another calls on teacher education.

Figure 14: Three graphs describing three different runs.

This task forced the students to interpret the graph, to make a table of suitable data points (time, distance), to mark the points at the floor, finding a way to keeping a step rhythm in accordance with suitable time intervals.

It did do great impression on students. They engaged eagerly in the activity. And it introduced an extra dimension to the work with functions as one student expressed it: 'Working with functions can even be fun'.

Oblique Throw dramatized

At a later stage of our work with parametric equations we touched the subject of oblique throwing with offset in a problem of a bouncing ball. Here it was also possible to use the video technique from [4] and compare the trajectory of the ball with parabolas. At my way home from the institute I passed by a ramp at the harbour in Aarhus and an idea to a task emerged. How fast should a car drive up the ramp to jump over some concrete pillars (left of the ramp on Fig. 15) that were placed in front of the ramp? Would it be realistic that some daredevil could manage to do it?

Figure 15: Ramp at the harbour, Aarhus, DK.

I only showed the students the photo and told them where the ramp was located considering it as a part of the game to go there and measure the relevant data for the calculations. Although I myself thought it to be an exciting challenge no students took the bait. Not that they didn't find it interesting (at least some of them) but they were too busy studying other subjects (than math) included in teacher education. They couldn't find the time. So they said. ☺.

Concluding remarks

Different set ups for working with coordinate systems and parametric equations was presented. Starting point for the students' work was a problem from an earlier written examination. Problems of the danger of ships colliding were the context given where the mathematical topics were embedded. This context seemed too far from the students' world of experience - at least in connection with learning mathematics - so they became confused. They found it much easier to solve a context free version of the problem.

As a teacher I wanted to change their attitude to working with mathematics in different contexts. Contrary to comply with the students wish for context free problems I introduced a number of new contexts for the same mathematical topic to give the students extra opportunities to become more familiar with this way of treating mathematics.

There were very different reactions from students to the introduction of more contexts. At the one end of the scale of reactions students were rather forthcoming and interested and found it refreshing and motivating even though they at the same time found it harder, more demanding and time consuming. The time to work it through was seen as an obstacle to do the work in a satisfying way.

At the other end of the scale a couple of students reacted almost hostile to the idea of introducing variation in contexts. Here the focus primarily was on passing the written examination. From earlier upper secondary school experience these student expected problems to be standard and solvable by standard methods. Training mathematical awareness in different context seemed to them going astray in time consuming and confusing activities that would not help them to pass examination. Their ideas about the challenges of their future career as teachers in the Danish municipal school seemed too far away and their conceptions about mathematics and teaching in mathematics were still dominated by beliefs established during upper secondary school.

For the author of this article as an educator of teachers for primary school it nevertheless will be important to challenge the students' beliefs about mathematics and the teaching of the subject. Therefore the introduction of lots of different contexts for the mathematical subjects in the curricula is here to stay as long as the curriculum for mathematics in the primary and lower secondary school as the very first aim contains the following (author's translation from Danish):

The aim of the teaching is that students develop mathematical skills and gain knowledge and skills so that they become able to cope adequately in math-related situations in daily life, social life and nature. [7]

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THE NORSE TREATISE ALGORISMUS

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ABSTRACT. *Algorismus* is a treatise on Hindu-Arabic numerals, composed in the 13th century and found in four manuscripts dated 1300–1550. It is a translation of *Carmen de Algorismo*, a hexameter by Alexander de Villa Dei. Here, additions in *Algorismus* to *Carmen* are examined and the four manuscripts are compared by a numerical method.

Algorismus

The thirteenth century treatise *Algorismus* has been preserved in manuscripts, written in the Norse language, spoken in Iceland and Norway in the Middle Ages. The bulk of the treatise is a prose translation of the Latin hexameter poem *Carmen de Algorismo*, written in France in the early thirteenth century. The poem and the treatise introduce the Hindu-Arabic number notation to Europeans, the treatise to Icelanders and Norwegians in particular.

Algorismus exists in four manuscripts, AM 544, 4to, AM 685, 4to, AM 736 III, 4to, preserved in Copenhagen, and GKS 1812, 4to, preserved in Reykjavík.

In this paper we explore incidences where *Algorismus* deviates from *Carmen de Algorismo* and compare the four extant manuscripts of *Algorismus* numerically.

Algorismus was first published in a scientific edition 1892–1896 by Finnur Jónsson. The basic manuscript used was AM 544, 4to, corrected using the three other manuscripts when applicable. The Norwegian mathematician Otto B. Bekken translated *Algorismus* into modern Norwegian in 1985 and explained its text in cooperation with linguist Marit Christoffersen (Bekken & Christoffersen, 1985). Kristín Bjarnadóttir (2002; 2004; 2007, pp. 43–47) has explained the content of *Algorismus* in English and modern Icelandic. Helgi Guðmundsson (1967) has conjectured that Algorismus was composed in Viðey monastery, on Viðey Island off Reykjavík.

The contents of Carmen de Algorismo and Algorismus

Algorismus contains an explanation of the Hindu-Arabic number notation, including its place value and seven arithmetic operations: addition, subtraction, doubling, halving, multiplication, division and extraction of roots: the square root and cubic root. These methods have been transferred to *Algorismus* via a well-known Latin hexameter, *Carmen de Algorismo*, written by the Frenchman Alexander de Villa Dei between 1200 and 1203 (Beaujouan, 1954, p. 106).

The manuscript MS. Auct. F.5.29, containing *Carmen de Algorismo*, preserved in the Bodleian Library in Oxford, dated to the thirteenth century, has been compared to *Algorismus* in the manuscript AM 544, 4to. The comparison reveals

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that *Algorismus* is a direct translation of *Carmen de Algorismo*, found in MS. Auct. F.5.29. Both manuscripts have chapter headings which are found neither in the other manuscripts of *Algorismus* nor in printed versions of *Carmen* (Steele, 1988; Halliwell, 1841).

Carmen de Algorismo is an extraction of Muhamed ibn-Musa al-Kwarizmi's *Kitab al-jam´val tafriq bi hisab al-Hind* (1992). *Carmen* was one of the first works introducing Hindu-Arabic number notation and arithmetic to Europeans. The poem exists in a great number of manuscripts, preserved in libraries in France, Great Britain, the Netherlands and many other countries, and is considered to have played a greater role in distributing Hindu-Arabic number notation in Northern Europe than the well-known *Liber abaci* by Leonardo da Pisa.

The arithmetic operations of addition, subraction and division, explained in *Carmen* and *Algorismus*, are largely similar to present methods used in paper-and-pencil arithmetic. Multiplying two composite numbers, however, proceeds from the left, as opposed to common modern algorithms. The numbers to be multiplied are arranged so that the digit farthest to the right of the multiplicand is placed below the first digit (from left) of the multiplier. The multiplicand is multiplied by this digit which then disappears under the product. Then the multiplicand is moved one place to the right so that the rightmost digit is placed below the second digit of the multiplier in the upper row and the lower number is now multiplied by this in the same manner. The product is written over the multiplying digit and added to the next digits to the left as before.

Carmen and *Algorismus* do not illustrate their algorithms on the four arithmetic operations by examples. The following example is made up for clarification in this paper:

Multiply 523 by 217:

First 523 is multiplied by 2 and 2 disappears under the product:

217	**1046**17
523	523

The first digit on the right of the lower number is then moved one place to the right below the second digit of the upper number which now is the multiplying digit. In this example the digit 3 is to be placed below 1 and other digits similarly. Now 523 is multiplied by 1. It is not quite clear if the digits of the product should be added as they are calculated or afterwards all at the same time:

52	
104637	**10983**7
523	523

In the next step the digit 3 of the multiplicand is moved below 7 and the other digits similarly. Then 523 is multiplied by 7:

	35		**14**	**2**	
109837	1**0983**7	**11333**7	**11333**7	**113471**	**113491**
523	523	523	523	523	523

The order of the multiplications in a product of two three digit numbers is as follows:

(100a + 10b +c) multiplied by
(100d + 10e + f)
= 100a·100d + 100a·10e + 100a·f + 10b·100d + 10b·10e + 10b·f + c·100d + c·10e + c·f

The advantage of multiplying from the left is that the product of the digits can be added to the previous product as they are found and it is not necessary to carry.

Deviations of Algorismus from Carmen de Algorismo

Carmen de Algorismo is a poem to be read aloud and thus a verbal work. The beginning of the MS. Auct. F.5.29 reads as follows:

Hec algorismus ars presens dicitur ; in qua
Talibus Indorum fruimus bis quinque figuris
0 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1 ·
Primaque significat unum : duo vero secunda :
Tercia significat tria : sic procede sinistre
Donec ad extremam venias, que cifra vocatur ;

The ten digits in the third line of the poem are the only incidence where the new Hindu-Arabic digits are presented in *Carmen*. Everywhere else numbers are expressed in words. The poem explains algorithms which are now common but it does not show any examples. It is not known how the poem was used as an aid to computation but one may assume that calculations were made on tablets or a flat surface, strewn with sand, or else on a wax tablet.

Figure 1: The ten digits of Hindu-Arabic numerals in Carmen de Algorismo in MS. Auct. F.5.29 (Bodleian Library).

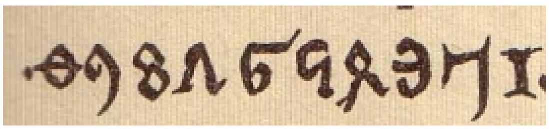

Figure 2: The ten digits of Hindu-Arabic numerals in Algorismus in AM 544, 4to (Jónsson, 1892–1896, p. 417).

Carmen de Algorismo is believed to be the first work where the zero, cifra, is presented as a digit (Beaujouan, 1954, p. 106).

The treatise *Algorismus* reacts to *Carmen's* lack of demonstration of the new system's number notation. It adds extensions in the first chapters, which point to a need to clarify the text by some numerical examples. Already after *Carmen* explains the place value notation, examples are inserted in the Icelandic translation.[1]

> Ergo, proposito numero tibi scribere, primo
> Respicias quis sit numerus ; que si digitus sit,
> Primo scribe loco digitum [**this way, 8**]; si compositus sit
> Primo scribe loco digitum post articulum sit [**as here, 65**]
> Articulus si sit, cifram post articulum sit [**such, 70**]
> (Jónsson, 1892–1896, p. 417).

Next even and odd numbers are presented, where the following addition is inserted in *Algorismus*:

> Quolibet in numero, si [**it is a multiple of ten or**] par sit prima figura,
> Par erit et totum, quicquid sibi continuetur;
> Impar si fuerit, totum sibi fiet et impar.

Furthermore *Algorismus* adds for explanation that

> **even digits are four, 2, 4, 6 and 8, and uneven [odd] are another four, 3, 5, 7, 9. But one is neither as it is the origin of number** (Jónsson, 1892–1896, p. 418).

The digits are written in Hindu-Arabic mode in all extant manuscripts. *Algorismus* adds that one is neither even nor odd number as it is the origin of all number. Bekken et al. (1985, p. 27) have pointed out likeness to the statement that one is not a number, in al-Kwarizmi's *Arithmetic*, which again refers to another book on arithmetic, most likely either Euclid's Elements, book VII, or *Arithmetica* by the Neo-Pythagorean Nicomachus. The citation referred to is the following from the translation *Dixit Algorizmi* of al-Kwarizmi's work:

> J'ai déjà expliqué dans le livre d'algebr et almucabalah, c'est-à-dire de reprise et de rejet, que tout nombre est composé et que tout nombre est formé sur l'unité. L'unité se trouve donc dans tout nombre. Et c'est ce qui est dit dans un autre livre d'arithmétique, que l'unité est l'origine de tout nombre et existe en dehors d'un nombre (Allard, 1992, p. 1);

> I have already explained in the book on algebra and almucabalah, that is on restoring and comparing, that every number is composite and every number is composed of the unit. The unit is therefore to be found in every number. And this is what is said in another book on arithmetic that the unit is the origin of all numbers and is outside numbers.

Next time *Algorismus* adds an example, is when Carmen's text states that there are seven operations: addition, subtraction, doubling, halving multiplication,

[1] In the following examples, the Latin text is taken from the Carmen-manuscript MS. Auct.F.5.29, while the Old Norse additions are taken from Jónsson's edition of Algorismus, and translated into English. Translations from the Old Norse language were made by the author, K. B.

division and root extraction. Then *Algorismus* states that root extraction has two forms, extracting square root and cubic root:

> ... and is that branch in two ways. One is to take a root from a square number but the other to take a root from an eight-vertex number, which grows cubically (Jónsson, 1892–1896, p. 418).

Another elaboration of al-Kwarizmi's work by Sacrobosco, *Algorismus Vulgaris*, says: "... extraction of roots, which is twofold, since [it applies] to square numbers and cube numbers." (Grant, p. 95). This quotation points to that the translator of text may have known Sacrobosco's text in addition to Villa Dei's *Carmen*. Sacrobosco claims, however, that there are nine species of arithmetic operations, adding numeration and progression as operations no. one and eight.

Each arithmetic operation is explained in a separate chapter. To multiply, the reader is instructed to arrange the two numbers to be multiplied in columns such that the first digit (from the right) of the multiplier is placed below the last digit of the multiplicand. However, first one must check the difference of the larger digit of the multiplicand from ten and then delete the smaller one from its tens as often as that difference:

> In digitum cures digitum si ducere, major
> In quantum distet a denis respice, debes
> Namque suo decuplo tociens delere minorem;

Algorismus adds this explanation:

> **So that you understand this multiply vii and nine. Nine differs by one from x. Therefore take one vij from vij tens. Then remain iij and vi tens, that is vij times nine** (Jónsson, 1892–1896, p. 420).

In modern notation this may be written

$$7 \cdot 9 = 10 \cdot 7 - 1 \cdot 7 = 63 \qquad \text{or}$$

$a \cdot b = 10a - (10 - b)a \quad (0 < a, b < 10)$.

Two conclusions may be drawn from this explanation. First, the Latin text is not considered to be clearly presented so that an example is needed. Second, the example demonstrates that as the translator/transcriber is not familiar with Hindu-Arabic digits, he uses Roman numerals and words. Numerals do not have a consistent representation across manuscripts, in the youngest manuscript some numbers are written in the Hindu-Arabic mode.

The multiplication example is the last one added for clarification in *Algorismus*, while several repetitions in *Carmen* were omitted in the translation.

Finally, a separate chapter is added to the translation on the cubic numbers 8 and 27 and their intermediate numbers 12 and 18, and their relation to the elements: Earth, $2^3 = 8$; Water, $2^2 \cdot 3 = 12$; Air, $2 \cdot 3^2 = 18$; Fire, $3^3 = 27$. This chapter does not exist in *Carmen*, and its content is unrelated to the main bulk of *Algorismus* in modern understanding. It says that it has been found necessary to add something in between the Earth and the Fire to unite them in their disagreement. Therefore, Water has two parts from Earth and one from Fire, and Air has one part from Earth and two from Fire. This puts the Elements in the correct order by lightness: Fire (27), Air (18), Water (12) and Earth (8) (Jónsson, 1892–1896, p. 423–424).

This produces the sesquialteral progression 8:12::12:18::18:27 or n:n+½n. This sequence of proportions and elements appears in St. John's College MS 17 (Oxford Digital Library), and in a similar schema in an eleventh century manuscript of Boethius, Madrid Biblioteca nacional Vit. 20 fol. 54v, and in a thirteenth c. Macrobius manuscript British Library Arundel 399 fol. 12v. It is also found in the anonymous treatise on cosmology in Bodleian Library Digby 83, fol. 3r (The Calendar and the Cloister – St. John's College MS 17, commentary; Bekken, 1986, p. 16).

The four manuscripts of Algorismus

The texts of *Algorismus* in the manuscripts AM 544, 4to, and GKS 1812, 4to, are identical in most respects, as is AM 685, 4to, which however has a 306 word long addition, not treated in this paper.

AM 544, 4to, is the oldest manuscript of the treatise, estimated to be written in the period 1302–1310, most likely in 1306–1308 (Karlsson, 1964*,* pp. 114–121). The text is divided into chapters which bear headings. Numbers are written using Hindu-Arabic numerals in the introduction and in the additions to *Carmen* by examples of place value notation and even and odd numbers shown earlier. Numbers are, however, mainly written by Roman numerals, until in the last section on the Elements, which does not belong to *Carmen de Algorismo*, and where Hindu-Arabic numerals are used.

The part of **GKS 1812, 4to**, containing *Algorismus* is estimated to be written in 1300–1400 (*A Dictionary of Old Norse Prose*, 1989– , p. 26). There are no chapter headings. Numbers are mainly written using words as in *Carmen de Algorismo*, exceptionally using Roman numerals and never using Hindu-Arabic numerals apart from in the first additions to *Carmen*, as is done in AM 544, 4to, and in the chapter on the Elements.

AM 685, 4to, is dated to 1450–1500 (A Dictionary of Old Norse Prose, 1989– , p. 26). It has no chapter headings. Number are written alternatively in words, Roman numerals and Hindu-Arabic notation, which is the most common. F. Jónsson states that the text of *Algorismus* in AM 685, 4to, is the most error free of the four texts, basing this conclusion on various spelling examples (Jónsson, 1892–1896, p. cxxxi). Furthermore this text is the most concise of the four texts as it is often contracted, preserving a correct meaning. In the following comparison the difference in spelling is not revealed as the texts of all the manuscripts have been rewritten to modern Icelandic. The text in AM 685, 4to, is also correct where other texts have an error on the origin of one half (Jónsson, 1892–1896, p. 419), called *semiss*, coming up after halving a number, which indicates that one of the transcribers of AM 685, 4to, understood the treatise well.

AM 736, 4to, is estimated to origin around 1550 (*A Dictionary of Old Norse Prose*, 1989–, p. 26). It contains only a fragment of the text of *Algorismus*, a section on root extraction. It does not contain the text on the Elements and their associated numbers, while on a different leaf in the same manuscript a diagram of the four Elements is found together with the Elements and the number xii associated to Water, xviii to the Air and *tres, trium, tres* to the Fire. Diagrams with the Elements and the four numbers exist in other manuscrips as cited earlier, but they are unrelated to *Algorismus* (Bekken, 1986, p. 16).

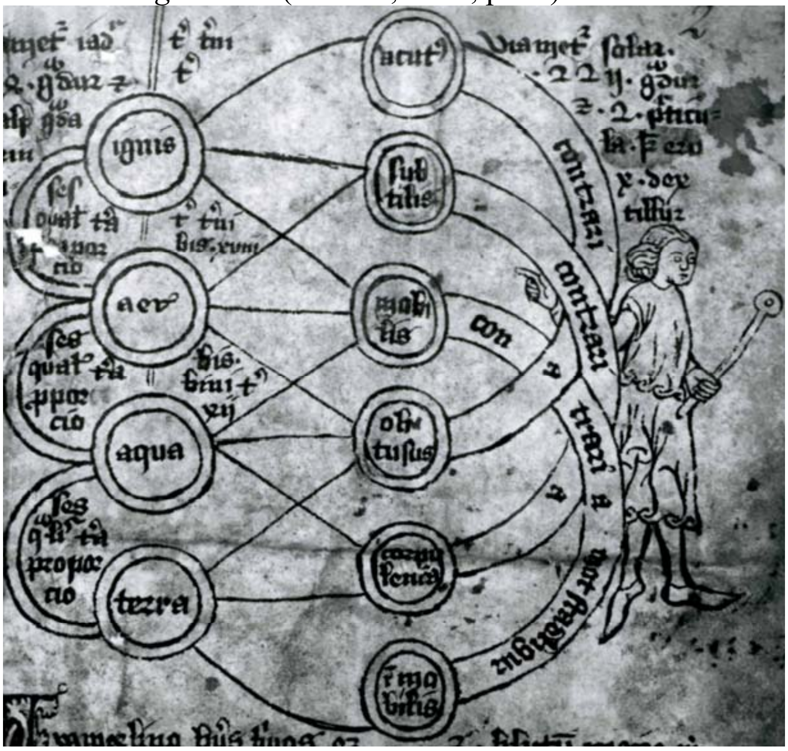

Figure 3. A diagram of the four Elements in the manuscript AM 736 III, 4to.

The adaptations made to *Carmen de Algorismo* to create *Algorismus* suggest that *Algorismus* served a role in introducing the use of Hindu-Arabic numerals in Iceland. In the oldest manuscript of *Algorismus*, AM 544, 4to, Roman numerals are used to explain the text or plain words as in *Carmen*. The use of Roman numerals suggests on one hand that the transcriber needed to shorten the text and on the other that he was not used to Hindu-Arabic numerals.

Plain words are dominant in GKS 1812, 4to. The youngest whole manuscript, AM 685, 4to, rarely has Roman numerals, while words and Hindu-Arabic numerals are used alternatively.

Manuscript comparison - methodology

When reading the four manuscripts of Algorismus it is apparent that they are quite similar; sentence structure and phrasing suggests that they all have the same root. The same text insertions and deletions are made in all four manuscripts to Carmen de Algorismo, exemplifying that these are not different translations. But how similar are these manuscripts?

Numerical methods were used to compare the manuscripts, comparable to methods that have been used extensively in comparative linguistics and in gene and protein comparison.

The four texts were aligned using the computer program ClustalW and a weighted number of mismatches between the manuscripts was computed. As ClustalW is designed to align protein sequences it takes as input sequences from the twenty letter alphabet of protein sequences. As the Latin alphabet is larger than twenty letters, each letter was mapped to two letters in the alphabet of protein sequences. ClustalW was then used to align the texts and the text was mapped back to the Latin alphabet. The alignment was then corrected manually, considering in particular word reorder and different forms of the imperative.

Mismatches between the manuscripts were counted and classified into three distinct classes; *single character mismatches*, *word reorders* and *word mismatches*. *Single character mismatches* were defined as:

- Spelling is identical apart from a single character difference.
- Mismatches in writing style of the numerals; Hindu-Arabic, Roman or spelled out.
- Mismatches in the writing of the imperative, e.g. tak þú - taktu.

Word reorders were defined as parts of the manuscripts where the order of two or more words had been reordered.

Word mismatches were all other types of differences such as word insertion, missing words or a different word being used.

The weighted distance between the manuscripts was used to infer the phylogeny of the manuscripts, using the assumption that it is unlikely that the same change is made more than once. One may also assume that each transcriber is equally likely to cause a distinction.

Finally, a simple program was written to count the number of differences.

Results

The manuscripts are different in length. In the following a section in AM 685, 4to, of length 306 words, not extant in the other manuscripts, has been removed. The lengths are:

Manuscript	Words #	Characters #
AM 685, 4to	2902	14772
AM 736 III, 4to	630	3323
AM 544, 4to	2960	15110
GKS 1812, 4to	2986	15174

Table 1. No. of words and characters in the four manuscripts of Algorismus.

That AM 685, 4to, has fewest words of the complete manuscripts confirms the reader's intuition that the transcriber(s) of AM 685, 4to, sometimes shorten the text.

The following weights of mismatches were used:

Word mismatches: 1,00 point

Word reorders: 0,25 point

Single character mismatches: 0,25 point

Results from counting mismatches between the three complete texts in AM 685, 4to, AM 544, 4to, and GKS 1812, 4to, were:

Manuscripts	AM 685, 4to	AM 544, 4to	GKS 1812, 4to
AM 685, 4to	0,00	261,00	264,50
AM 544, 4to	261,00	0,00	123,25
GKS 1812, 4to	264,50	123,25	0,00

Table 2. No. of mismatches between the three complete manuscripts of Algorismus

The greatest distance between two manuscripts is between AM 685, 4to, and GKS 1812, 4to, 264,5 mismatches by 2986 words, or 8,9%.

The shortest distance between two manuscripts is between AM 544 4to, and GKS 1812, 4to, 123,25 mismatches by 2986 words, or 4,1%.

The parts of the manuscripts that they have all in common, that is the part also found in AM 736 III, 4to, were compared separately. The results were:

Manuscripts	AM 685, 4to	AM 736 III, 4to	AM 544, 4to	GKS 1812, 4to
AM 685, 4to	0,00	73,25	52,00	55,00
AM 736 III, 4to	73,25	0,00	57,25	57,75
AM 544, 4to	52,00	57,25	0,00	26,00
GKS 1812, 4to	55,00	57,75	26,00	0,00

Table 3. No. of mismatches in the part common to all manuscripts of Algorismus

The distance of AM 736 III, 4to, is greatest from AM 685, 4to, while it is closest to AM 544, 4to, and nearly equally close to GKS 1812, 4to.

Clearly, AM 544, 4to, and GKS 1812, 4to, are more close to each other than the other two, which are also different from each other.

In this counting of mismatches the ratio 1 : 0,25 or 4 : 1 between *word mismatches* and other mismatches was used. Counting was also done using the ratio 3 : 1 and lead to comparable conclusions.

Figure 3 exhibits the relation beween the different copies of *Algorismus*. A matrix was made according to the distances between the four manuscripts, from which was constructed a phylogenetic tree with distances similar to the distances in the distance matrix. The diagram was made by the program ATV (Zmasek and Eddy, 2001, p. 383 – 384).

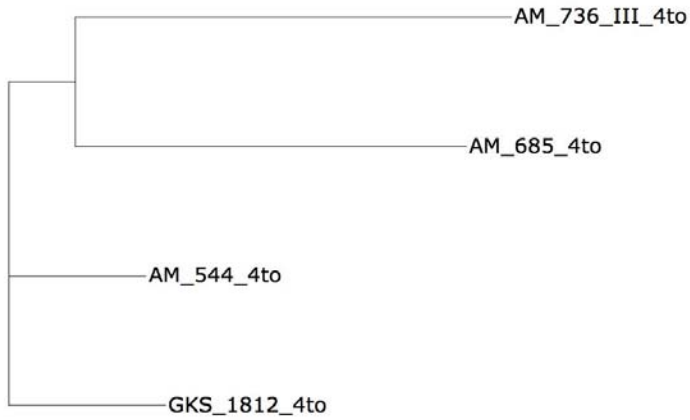

Figure 4. A phylogeny of the copies of Algorismus in the manuscripts AM 736 III, 4to, AM 685, 4to, AM 544, 4to, and GKS 1812, 4to, made by the program ATV.

The phylogeny may be interpreted such that the manuscript AM 544, 4to, contains the most original copy of the treatise, and that the copy in GKS 1812, 4to, is closest to it. The copies in the manuscripts AM 736 III, 4to, and AM 685, 4to, are partly drawn from the same stem, but are further from the origin, in particular AM 736 III, 4to.

Discussion

What motivated Icelanders to translate Carmen? Certainly, they had to count their belongings and assets. e.g. for taxes, but they could have done that with the Roman numerals they knew. Writing manuscripts was an integral part of the Christian monastic culture. The reason may have been an aspiration to belong to the European cultural world, while Latin was an alien language to the recently literate Germanic tribes that had settled in the Nordic countries. The population in Iceland was never large, around 50–70,000 in the 13th century. Producing writings in the vernacular was an important factor in creating a common culture of this small group of people.

The additions in Algorismus to its original, Carmen de Algorismo, bear witness to a desire for learning, to understand the text. Comparison of the four copies of Algorismus of different age reveals that people continued to work on understanding the text and gradually began to use the convenient Hindu-Arabic number notation.

But it took time. According to the phylogeny and other considerations, manuscript AM 544, 4to, was not the original of Algorismus, which suggests that Algorismus may have been written in the second half of the thirteenth century, or about 200 years before AM 685, 4to, and possibly up to 300 years before AM 736 III, 4to. Algorismus therefore played an important role in Icelandic culture until the era of printing, when printed books began to spread much more rapidly between countries than manuscripts.

Iceland was originally an independent society, but from 1397 it belonged to the Danish realm until 1944. Due to worsening living conditions and its colonial status, Iceland lagged behind other European countries in educational respects. Algorismus appears in history whenever mathematics education was revived, serving as a monument of the proud past, when Icelanders kept up with the latest global knowledge. Even the most distinguished Icelandic scholars continued to refer to Algorismus up until the nineteenth century, (see e.g. Gunnlaugsson, 1865, p. 4), paying respect to the time when Icelanders were familiar with the latest mathematical knowledge in the world and translated it to their own language.

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Manuscrips

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AM 736 III, 4to, 2r, 4r–4v.
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MS. Auct. F.5.29.
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The Calendar and the Cloister – St. John's College MS 17.
http://digital.library.mcgill.ca/ms-17/

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ENHANCING THE RESEARCH POTENTIAL OF MATHEMATICALLY GIFTED HIGH-SCHOOL STUDENTS

NELI DIMITROVA, OLEG MUSHKAROV, EVGENIA SENDOVA

ABSTRACT. The authors share their experience in the context of two programs for enhancing the potential of gifted high-school students to do research in mathematics and science - the Research Science Institute (RSI) in US, and the High-School Science Institute (HSSI) in Bulgaria. The focus is on the ways the mathematically gifted students are supported through their research in the context of RSI and HSSI. The specific role of the mathematics mentors and tutors is discussed together with some examples of topics of math projects.

1 Introduction

It is hardly a paradox that one could love mathematics, admire its beauty, and still be put down as an eccentric with a strange hobby since the general public of today is under the impression that mathematics is mainly the art of doing sums. One of the reasons many students come to view mathematics as a static body of facts and knowledge divorced from their world of experience is that fewer than 1% of all mathematics concept they learn were discovered after the 18th century [1]. Establishing connections between the students' study of mathematics and the work of the current mathematicians can *breathe life* into the mathematics classroom. Not only should the students be given the chance of seeing the real nature of mathematics at school age but they can begin *doing mathematics* in their schooldays. There is no doubt that *being a mathematician, like being a poet, or a composer or an engineer, means doing, rather than knowing or understanding* [2].

In this paper we concentrate on some approaches of working with high school students in the frames of specially designed research programs and we share our experience in national and international context. We pay special attention to the role of working on research projects in enhancing students' math competence and motivation to study mathematics or related to it fields.

2 Conveying the inquiry based spirit of mathematics at school age

To appreciate the real beauty and meaning of mathematics as a scientific field and possibly choose it as their future profession the students should be enabled to participate in some forms in which: to use mathematics in daily life activities; to apply mathematical thinking and modeling so as to solve problems that arise in other fields; to use mathematical methods as an integrated whole; to formulate their own hypotheses and problems, and to attack open problems. For students at school age to experience at least partially these sides of the math research process various forms exist such as: specialized research programs, school sections in the frames of professional conferences, symposia and fairs for young scientists. Many researchers in gifted education believe that educational programs outside of schools are absolutely necessary for gifted children because they meet their special learning needs by providing more opportunities for

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program is a great chance to meet other bright kids who are fascinated by learning. These are their true intellectual peers. Courses in these programs combine the best of both worlds: accelerated content and bright age-peers [4]. Summer programs vary in terms of content, duration, intensity, sponsorship, and overall purpose. Still some general benefits are found to include the following:

- Perceptions of increased social support for learning and achievements due to homogeneous grouping and support from counselors, tutors, and mentors;
- Positive feeling resulting from a more appropriate match between the student's academic potential and the challenge of the research projects;
- Development of skills for intensive study and for doing scientific research;
- Reinforcement for risk taking as a result of extending oneself intellectually and socially;
- Growth in acceptance of others and (in the case of international component) knowledge of different cultures.

Below we shall describe the infrastructure of two research programs in which the authors have been involved − the Research Science Institute (RSI) held at the Massachusetts Institute of Technology (MIT, US), and the Bulgarian High School Students Institute of Mathematics and Informatics (HSSI).

3 Good practices in an international context − the RSI summer program

The RSI was developed by the Center of Excellence in Education (CEE), a non-profit educational foundation in McLean, Virginia. The Center was founded by the late Admiral Rickover and Joann DiGennaro in 1983, with the express purpose of nurturing young scholars to careers of excellence and leadership in science, mathematics, and technology. Central to CEE is the principle that talent in science and math fulfills its promise when it is nurtured from an early age. RSI is an intensive annual six-week summer program which is sponsored jointly by CEE and MIT. It is attended by approximately 80 high-school students from US and other nations including Bulgaria, China, France, Germany, Greece, Hungary, Israel, Lebanon, Poland, Singapore, Saudi Arabia, and the UK. Once selected, the students go to MIT and work on a research project under the guidance of faculty, post-docs, and graduate students at MIT, Harvard, Boston University, and other research institutions from Boston area. All the students chosen for the RSI have already acquired a deep interest in a scientific field − mathematics, CS and natural sciences. The Institute begins with four days of formal classes. Professors of mathematics, biology, chemistry and physics give lectures on important aspects of their field and their own research. The students also attend evening lectures in science, philosophy, and humanities delivered by eminent researchers including Nobel Prize winners. The internships that follow the first week classes comprise the main component of the RSI. Students work in their mentors' research laboratories for five weeks. At the end of the internship they present a paper on their research and give an oral presentation in front of a large audience at the RSI Symposium. It is best to characterize the RSI research paper as a progress report for a continuing research effort − the students write progressive versions of the paper and prepare presentations of their work throughout the program using a consistent intellectual template to which the tutoring staff can target their support. Progress reports typically focus more on methods and process than a final research paper but they naturally evolve into final reports as some original results are obtained. The transition from progress report to final research paper is typically one of reduction through editing of existing text with the perspective of the final results in mind. RSI is well structured for this reduction process as last week teaching assistants and the so called nobodies (RSI alumni with no formal duties) supply great quantities of quality editing advice in the week before the papers are due. Especially important in the process of preparation are the milestones − intermediate steps of the process. Typical milestones for the written presentation are: writing about a mini-project using the same sample as the one for the final paper; gradually filling the proposed sample starting with the background of the project, the literature studied and the methods used; considering partial cases and possible generalizations; classifying the cases of failure, etc. Possible milestones for the oral presentation are: speaking for three minutes on a freely chosen topic, presenting the introductory part of the project for five minutes, etc. All the milestones are accompanied by a feedback from the tutors who work closely with the students − they read and critique the draft papers, provide editorial remarks, suggest avenues of research and areas of additional background reading, give ideas for improvement of the oral presentations, etc. Both, the written and the oral presentations are the culmination of students' research in mathematics, physics, chemistry, biology, computer science, and engineering. The RSI faculty together with a team of outside experts selects five outstanding written reports for special recognition. A panel of visiting educators and scientists also choose five projects to receive awards for outstanding oral presentation. The winning written reports are printed first in their entirety, followed by the abstracts for the reports that have won the oral-presentation awards, and finally by the remaining student abstracts. The compendiums of three consecutive years [5] provides an idea of the variety of topics of projects performed at RSI.

4 A Bulgarian experience of mathematics research at school age – the HSSI

The long-term collaboration between the Center for Excellence in Education, Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences (IMI-BAS) and *St. Cyril and St. Methodius Foundation* was one of the crucial factors for the establishment of the *High School Students Institute of Mathematics and Informatics* (HSSI) in the World Mathematical Year 2000 [6]. Founders of HSSI are the Union of Bulgarian Mathematicians (UBM), the Foundation *Evrika*, the International Foundation *St Cyril and St Methodius* and IMI-BAS. One of the main ideas behind HSSI was to implement RSI-like activities in Bulgaria, taking into account the local conditions and traditions. This institute inherited the good traditions of an earlier movement of the technically creative youth in Bulgaria.

The local conditions included the infrastructure and activities of the UBM which has long-standing traditions in early identification and proper enhancement of talents. Since 1980, School sections in the framework of the annual Spring Conferences of UBM have been organized. This contributed naturally to the mission of HSSI to keep the traditions alive giving them new spirit and new content. Another important component of the local conditions is the environment provided by the IMI-BAS stimulating the growth and the progress of HSSI − library, Internet, rooms and equipment. Many researchers at the Institute devote significant part of their free time to keep the level of extra-curricular work with talented students high [7]. Their work supports and enables HSSI to assist the intellectual and professional growth of the high school students. The participants in HSSI are high school students between 8^{th} and 12^{th} grade, mainly from specialized science and mathematics secondary schools. Every participant in HSSI works individually (or in a team) on a freely chosen topic in mathematics, informatics and/or Information Technologies (IT) under the guidance of a teacher or another specialist. A written presentation of the project is sent to HSSI. All papers are refereed by specialists and the reviews are given to the authors. Papers involving creativity elements are given special credit. The best projects are accepted for presentation in the conference sessions of HSSI.

HSSI organizes three events per year − two conference sessions and a research summer school (RSS). The High School Students' Conference is held in January at the Plovdiv branch of the HSSI (the Faculty of Mathematics and Informatics at Plovdiv University) and is usually attended by more than 200 students, teachers, researchers in mathematics and informatics, parents, journalists. The conference is held in two streams − mathematics and informatics/IT. The authors present their work in front of a Jury of specialists in the field and in the presence of a general audience. The jury can ask the students various questions so as to check the level of their understanding and creativity. Parallel to these two streams is a poster session. Based on the merits of the paper and the style of presentation, the Jury selects the best ones. Their authors receive Certificates for excellence and are invited for an interview for selecting two Bulgarian participants in RSI and to participate in the School Section of the Spring Conference of UBM. The School Section is an independent event − it could be attended by students who present their research for a first time. The process of reviewing and selecting papers for the School Section is the same as above. The authors of the best projects from this section are invited to participate in the Research Summer School. One of the most important components of HSSI is the three-week Research Summer School. The RSS takes place in July-August in Varna and Uzana. During the first two weeks, lectures and seminars in mathematics are delivered by eminent specialists from universities, academic institutions and software companies. The main goal of the training is to extend the students' knowledge in topics related to their interests and to offer new problems to be studied and solved in further projects. Since 2005, the emphasis is on the development of short-term project under the guidance of mentors. The third week is devoted to a Students' workshop, where the participants report on their results and share ideas for further studies. To help teachers improve their mentoring skills a Teachers workshop is also organized during the third week of the RSS. The participants are the mentors of the students' projects, presented at the events of HSSI during the school year. Another important activity of the HSSI is its monthly seminar at the Institute of Mathematics and Informatics. The aim of the Seminar is to bring together high school students, teachers and scientists to present and discuss problems of common interest. The 10-year activities of the HSSI have been recognized in the frames of the European projects *Meeting in Mathematics* [8], *Math2Earth* [9], *InnoMathEd* and *Fibonacci* [10].

5 A math research project – *flying high and back to Earth*

Working on projects develops naturally skills important in life of the professional mathematicians − planning, searching for and selecting appropriate information, integrating knowledge from different fields (including informatics and IT), working in a team, presenting the results to a specialized and larger audience.

Our experience shows that the activities could be successfully grouped in several phases [11]:

Preparation phase − motivating the students for exploring a topic of interest by delivering short lectures and appropriate warm-up problems;

Research phase − engaging the students in research activities by formulating appropriate:

- short-term projects (expected to be developed for at most two weeks during a summer school or during the school year)
- long-term projects (lasting from 6 weeks to 4-5 months, in some cases − up to 2 years);

Extending the project with the formulation of open problems;

Presentation phase − building up ICT enhanced skills for a written and oral presentation of the project;

Passing on the torch − teaching students to act like mentors.

Let us illustrate some essential aspects of the students' math research and presentation.

Example 1: The strength of a metaphor for *bringing abstract mathematics concepts to Earth*

One of the most difficult things concerning the oral presentation of the projects is how to convey deep mathematical ideas for ten minutes to a larger audience of young scientists not necessarily mathematicians but eager to understand. Finding an appropriate metaphor is very helpful as the following extract of an oral presentation on hyperbolic geometry project shows:

Bryant Mathews (RSI'97)**:** *The idea came from a conversation I had with my mentor, Ioanid Rosu. When we say* geometry*, we could mean almost anything. In order to specify exactly what we mean, we must define certain objects and how these objects are related. In geometry, the two basic objects are the point and the line. Two axioms which we want these to satisfy are as follows:*

1) Two points lie on a unique line;

2) Between any two points on a line, there is another point which lies on the same line.

We usually think of points as dots on a chalkboard, and lines as curves connecting these points. But what if we define these objects differently? Suppose we define a point to be an ant, and a line to be a colony of ants. A point lies on *a certain line iff an ant is a member of a certain colony. Does this geometry satisfy the first axiom above? No, because two points need not lie on a line, i.e. two ants need not be in the same colony.*

Let us try again. Suppose a point is an ant, and a line is a pair of ants. Now the first axiom is satisfied, because any two ants form a pair of ants. However, there is never a third ant on the same line, so the second axiom is not satisfied.

Thus, our two axioms rule out each of the ant geometries; however, many geometries remain. As we add more axioms, we rule out more and more geometries until we are left with only one. If we choose the axioms of Euclidean geometry, we are left with our common notions of point and line. If we choose a different set of axioms, we may be left, for example, with the mysterious hyperbolic geometry, where rectangles do not exist.

Twelve years later Bryant's students express their admiration for the clarity with which he presents very abstract mathematical ideas [12].

Example 2: From a classical to an open problem

The most important goal of HSSI is to enable the students with special interests in mathematics to experience the spirit of the research process. This process doesn't end with finding the answer of a specific problem but often with posing new problems. Here is an example.

The second author: *Twenty years ago I came across the following problem formulated by Malfatti in 1803: Cut three circles from a given triangle so that the sum of their areas is maximal. In spite of the great interest this seemingly easy problem had raised, it was* *solved two centuries later by V. Zalgaller and G. Loss [13]. In conversations with students and teachers I have shared my opinion that the Malfatti problem is very suitable for a student research project since it provides rich opportunities for explorations of different variations of it. Three years ago, Emil Kostadinov (a student of the Mathematics High school in Blagoevgrad) presented a project on this topic [8, p. 165] which attracted my attention mostly with the authors' comments in the concluding part. Here they are in a nutshell. It follows from the theorem of Zalgaller and Loss that the solution of the Malfatti problem could be obtained by using the so-called greedy algorithm, viz. at each step we cut the largest possible circle. In Emil's project it is proved that the same is also true for the Malfatti problem in the case of square. Of course, one can formulate various Malfatti type problems and it is tempting to conjecture that their solutions could also be obtained by using the greedy algorithm. However Emil shows that this is not true in general. To do this he considers the following analog of the Malfatti problem: Cut three nonintersecting triangles from a given circle so that the sum of their areas is maximal. In this case the greedy algorithm does not lead to a solution as it can be derived by using the well-known fact that among all the n-gons inscribed in a circle the regular ones have maximal area. So it is natural to conjecture that the solution of the latter problem is given by three triangles forming a regular pentagon, inscribed in the given circle. To the best of my knowledge Emil's conjecture is still an open problem and I think it is a nice challenge to the students who like combinatorial geometry.*

Example 3: Getting the best from what is seemingly the worst

The problem with students already having certain experience in doing math research is that they expect to be given well formulated problem and appropriate resources from the very beginning of the five-week research period (even earlier if possible). When this is not the case there might be a great deal of disappointment, and even frustration. Here is the story of a Bulgarian participant in both HSSI and RSI:

Rafael Rafailov *(HSSI 2010 and RSI 2010) I was very happy when I learned that I was selected to participate in RSI, as I have always wanted to work with real mathematicians on an actual problem.*

My first choice for a research field for my participation in the program was topology, as this was a new area for me, which seemed interesting and exciting, and I wanted to do something different from what I had done before. I spent much time learning algebraic topology on my own before the beginning of the program and I was very excited to begin working on a real problem. After all my preparation I can't say I wasn't disappointed when I found out that the project I was supposed to work on for the duration of the program had nothing to do with what I had requested. However, I tried to get over my initial disappointment and tried to get the most I could out of this problem. Due to many different reasons things didn't quite work out and the problem I was working on was changed two weeks before the end of the program.

The final project I worked on was to extend a result in a paper by Prof. Curtis McMullen considering the Hausdorff dimension of a specific set of points on the complex which is invariant under transformation of the circle to itself. In the time of the program I wasn't able to completely prove a conjecture made by Prof. McMullen, but I was able to prove an upper bound on the Hausdorff dimension of the set and I am still working on proving a lower bound.

I think that RSI was a great experience for me, as even all the problems I came across, I had the chance to work on a problem proposed by a mathematician of the rank of Prof. McMullen, and gave me a problem to work on in the future. The RSI program also taught *me that things do not always go the way you expect them to go, but that one should always try to get the maximum out of every situation and not regret about anything, a lesson I think would be very useful future.*

6 The art of applying mathematics – order and chaos in population biology

An essential part of working with the students in the frames of HSSI is to demonstrate the art of applying mathematics in other fields of science. In our case, biology was the most natural field since the first author is specialist in biomathematics and not only did she introduce the ideas behind mathematical modeling but shared with the students that even the simplest mathematical models of bio phenomena could lead to deep mathematical problems with non-standard solutions. A summary of the cycle of lectures she offered to her students is presented in the article "*Order and chaos in a model of population biology*" [9, pp. 49-54]. Here follows an abstract of it:

In the 19th century, the Belgian mathematician *P. F. Verhulst* (1804-1849) devised a model that attempted to explain the increase in numbers of a population of creatures. It turns out that the population increase occurs predictably to a certain point, at which time the growth in numbers becomes chaotic. Although he did not realize it at the time, Verhulst's attempt to understand this behavior touched on chaos and fractals.

In order to discover some of the mysteries of a chaotic system, described by Verhulst's model of population growth let us consider the logistic discrete equation:

$x_{n+1} = f_r(x_n),\ n=0,1,2,\dots,$ where $f_r(x) = r\,x\,(1 - x),\ 0 < r \le 4,\ 0 \le x \le 1;$

n is interpreted as growth rate time, measured in discrete units (years, days, etc.), x_n is the number of species at time n, r is a parameter. As a rule, after a mathematical model is built, it is used to generate predictions: given some initial population size x_0 what can we say about the behavior of the orbit $x_0, x_1, x_2,\dots, x_n,\dots$ for large n? It is shown in the paper that for different values of the parameter r, orbits may have very complicated structure like convergence to a stable fixed point or to a periodic point or to other attracting sets; they may also show irregular (unpredictable or chaotic) behavior.

In addition to the introduction of some specific notions used in the theory of dynamical systems, the article proposes a variety of tasks, that can be used in mathematics and in informatics classes:

- Building composition of functions of the form $f_r(f_r(x)), f_r(f_r(f_r(x)))$ etc;
- Solving quadratic equations involving the parameter r;
- Finding derivatives of $f_r(f_r(x)), f_r(f_r(f_r(x)))$ etc;
- Exploring convergence of sequences.
- Using appropriate software (like computer algebra systems Maple, Mathematica) to visualize graphs of functions for different r
- Writing programs to compute orbits of the model and to visualize them

As seen the goal of this article is to introduce some of the most intriguing ideas of the dynamical system theory so as to motivate the readers to learn what to look for in their own studies of the dynamics in the world and the universe.

7 Conclusions

With an attitude to the teaching of mathematics, when students are encouraged to work in the spirit of discovery, they come to see mathematics in a new way—as an area in which interesting experiments can be made and hypotheses formulated. Even if they happen to reinvent the wheel, the students may feel the joy of the process of invention itself and acquire habits of creative thinking.

The best reward for all those involved in nurturing and developing the research potential of young mathematicians while they are still at school is to see various forms of collaboration among students, teachers and researchers. In our recent experience within HSSI and RSI we are satisfied to see joint publications having emerged from students' research in the frames of these programs ([14, pp. 135_140, 141_146, 151_157], [15]), and the effect of passing the torch to the next generation in new roles _mentors, lecturers, advisors. Some promising examples include HSSI projects mentored by Vesselin Dimitrov (Harvard), Alexander Lishkov (Princeton), Todor Bilarev (Bremen University), Konstantin Delchev (Sofia University), Nikolay Dimitrov (Sofia University), Antoni Rangachev (MIT). Thus, the effect of enabling students to experience the research spirit of mathematics at school age is not only to attract young people to enrolment in this field but also to educate and prepare them in becoming good mentors and ambassadors for mathematics.

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THINKING TOGETHER IN AND OUT OF THE SCHOOL VERSUS MATHEMATICAL RESEARCH

VLADIMIR GEORGIEV, VENETA NEDYALKOVA

ABSTRACT. We consider results of problem-posing and problem-solving projects developed by mathematical labs formed by teachers and pupils in High schools. Their work was supported and monitored by University researchers. The main impact of the projects and activities described below is the pupil's and teacher's motivation of learning mathematics.

Introduction

The idea to bring mathematicians and mathematical researchers into direct contact with High School teachers and pupils is not new. These motivated pupils and their teachers can meet with mathematics professionals in an informal setting, after school or on weekends, to work on interesting problems or topics in mathematics. These interactions get students excited about mathematics and seem to provide them further motivation to study deeper relations and branches of mathematics. The above idea can be used to compensate the well - known fact: studies show a lack in students' motivation in learning mathematics. This has a negative impact on several areas:

- Many fields in the world of work rely on an understanding of mathematics.
- Many decision-making processes in society, politics, economy, ecology, personal environment etc. require at least a basic understanding of principles in mathematics.
- Students who are less motivated to learn mathematics are much less likely to become motivated teachers in mathematics.
- On the basis of the above observation, a Comenius project "MATH2EARTH" started in 2008 and aimed at the following key points:
- Show that mathematics is everywhere, applications of mathematics in the real world. Particularly create modules to be used by teachers and students in school, and teacher trainers and trainees.
- Show that you can take (mathematically) important tasks and "dress them up", i.e. give them a context or a several possible contexts making it interesting for a variety of students. Again show some examples and also show how to open up a problem.

There are two possible choices to work in these direction: develop arguments and mathematical tasks in and out-of-school. The idea to create and "improve the sides" of the "triangle" University Researchers - High School Teachers - High School pupils suggests us to study the possibilities for out-of-school mathematical practice.

On the other hand, in general it is not so easy to realize and develop out - of - school mathematical practice. Most research shows a strong discontinuity between school and out-of-school mathematical practice. According to early work on situated cognition, e.g. Lave (1988), this discontinuity is a consequence of learning in and out of school being two distinct social practices. School mathematics, moreover, is often not suited to out-of-school practices: in some cases out-of-school problems are only apparently similar to

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school mathematics problems, but in reality there is a range of explicit and implicit restrictions which makes school methods unsuitable, and thus other methods are used (Masingila et al., 1996); in other cases (Scribner, 1984) work mathematics may appear to be simple, but there are no simple algorithms or methods to solve the problem and school-learnt procedures are of no use. As a conclusion there is a lack of best out-of-school mathematical practices.

In this article we shall try to give concrete examples how looking around us and trying to model the world around us will give a possibility to prepare and improve math capabilities of pupils, to introduce them to the beauty and excitement of mathematics. Natural inspiration for the Pisa team in the Comenius project was the fact that 2009 was the International Year of Astronomy that marks the 400th anniversary of the Galileo Galilei works. The legendary Leaning Tower of Pisa experiment by Galileo shows unlimited possibilities of the human mind to "see" and understand what is around us.

Obviously, there is an essential gap between fundamental discoveries of the Renaissance period and the standard math teaching in High Schools in Italy and Europe now. Somehow our attempt is to show some concrete examples how secondary school teachers in collaboration with University researchers can activate pupils creativity and see and recognize around them nice math (and not only math) problems.

The History keeps for us the legend of the algebraic battles of Niccolo` Tartaglia and Antonio Maria Fior, as well as, the competition between Lodovico Ferrari and Tartaglia. The problems, carefully chosen and presented to a Notary, have been printed and then distributed among scientists and researchers. Those works, not only brought immortality to those mathematicians, but also had a great influence on the development of the science.

In attempt to continue this noble tradition, we continue to invite and challenge small groups of secondary school math teachers and motivated students to postulate, solve and develop math arguments closely connected with the 400th anniversary of the Galileo Galilei and more precisely with Astronomy and Navigation. Our scope was not searching for great scientific results or young mathematical stars (we are proud to find any). We just want to involve the students in creating and solving problems, to invite them to share the joy and beauty of math thinking, to feel the excitement of discovering new results starting to look with open eyes around them. We are convinced that the wealth, contained in this part of human knowledge, must belong to all, not only a few chosen.

Methodology applied in the project

We are convinced that out-of-school activities and the collaboration between University researchers and future or in-service teachers have to be based on the following points:

a) application of a group method of teaching and studying: small groups (that we call math labs) of secondary school teachers and pupils work together and possibly use effectively Internet capabilities
b) work on concrete subject (in our case arguments closely connected with the 400th anniversary of the Galileo Galilei and more precisely with Astronomy and Navigation) and trying to see the world around us and prepare math models and math problems on the subject
c) possible interaction between these activities making part of the Extra Curricula activities and the standard Curricula ones. We are aware of the difficulties of this

interaction, but any small success here can effectively enlarge the group of motivated pupils (and even teachers).

Math labs or groups formed by math teachers and their students in High Schools started to work after 2005 in Tuscany, Italy in connection with the former Comenius project "Meeting in Mathematics". Let us mention that math labs are not typical for the Italian education system.

The purpose of this approach is to improve the relations between Secondary Schools and Universities in Tuscany to propose new attractive models of work with pupils from the Secondary Schools, improving the quality of math teaching.

To have a larger group of teachers and pupils involved we organized the out-of-school activity in such a way that it is implemented inside larger and interdiciplinary activity called "UnSungHero".

Let us make a brief description and explain the origin of this interesting phenomena and how it is connected with natural impacts as motivation of High schools pupils in mathematics.

The idea was generated in the publishing office of “Ulysses”, the magazine of the Pisan High School “Ulisse Dini”, by a teacher and a few pupils: their aim was to sort out some characters who could be taken as models, different from the usual ones. In other words, heroes with “intellectual” strength and meaningful lives. Young people are usually said to live with intensity the present, in order to bury the anxiety rising every time the future in sight appears waste and meaningless. The UnSungHero will give back the young a reason for living and lots of meaningful sights; the event wants to propose new models of life without any border of any kind. Short evaluation of the idea is given by Prof. Modica, mathematician and former chancellor of Pisa University.

“Many people might think it’s impossible to happen in the Italian school which is experiencing insufficient resources and a very unfavourable public opinion, particularly nowadays, when a lot of attention is paid to the negative information coming from the school world. But real school is something different: it does exist, though not everywhere; there are many pearls in it, shut in their own shells, which are its motive power, often without anybody noticing.”

The founder of the activity is a teacher in philosophy - prof. Rosana Prato from the same High School “Ulisse Dini”. The UnSungHero, elected by a jury of Italian and foreign students who take part in this project, is going to win a 5000 euro prize offered by Pisa Mayor. But in the UnSungHero edition 2006, a new initiative was born: the “Three days’ competition”. This is a challenge between the young participants of the UnSungHero. In the three days immediately preceding the final evening of the event, international and Italian pupils (or classes) face each other in different challenges organized by the University of Pisa and by other important cultural institutions. Winners get “special” prizes that consist of training opportunities. For example, in the 2005/2006 – 2006/2007 edition a class of a Brescia High School, that won the maths competition organized by the Maths Department of Pisa University, attended a three days’ intensive course at the Livorno Naval Academy. Later on, this course was followed by a three days’ practical course on the Amerigo Vespucci sailing ship.

Figure 1. Winners of math competition make a trip with the beautiful "Amerigo Vespucci".

Our next step is to describe how the math competition is organized.In the first phase of the math activity (called "Galilei" in the 2009 Edition of the UnSungHero) math labs were formed by groups having at least 5-7 pupils and at least one teacher in mathematics from the same Secondary School. After appropriate subscription the teams (mathematical labs) from Italy, Bulgaria, Russia, Japan proposed different mathematical problems.The teams were invited to prepare problems that are coming from the real life. Of special interest were the fields of navigation and astronomy. There was no need to know the solution of the problem posed. The approved problems were published and after this the teams had sufficient period of 1 month to prepare and send the solutions under the guide of their tutor/tutors - the math teachers in their math lab. To have a better and more precise comparison of the mathematical preparation of the labs an Internet competition (with 10 problems that has to be solved for 5 hours) was organized. On the basis of evaluation (with appropriate International Commission) of the problem - posing and solving work of the labs together with the results of the Internet competition a final score determined the first two teams (labs) that were invited to visit Pisa and have a direct final tour in Pisa. The final competition took place at the Department of Mathematics, University of Pisa, during the Three Days of the UnSungHero in June 2009.

It is important to notice that the competitiveness was not the dominant element in the organization of the work of the math labs. The key point evaluated in the work of math labs were creativity and novelty of mathematical ideas and the capacity to see deep mathematical problems in real life around us.

Results: astronomy and geometry on simplest curved manifold - the sphere

Spherical trigonometry is closely connected with the astronomy. Today the study of astronomy requires a deep understanding of mathematics and physics. It is important to realise that Greek astronomy (in the period of 1000 years between 700 BC and 300 AD) did not involve physics. Indeed, a Greek astronomer aimed only to describe the heavens while a Greek physicist sought out physical truth. Mathematics provided the means of description, so astronomy during this period of 1000 years was one of the branches of mathematics. An essential development which was absolutely necessary for progress in astronomy took place in geometry. Spherical geometry was developed by a number of mathematicians with an important text being written by Autolycus in Athens around 330 BC. Some claim that Autolycus based his work on spherical geometry On the Moving Sphere on an earlier work by Eudoxus. Whether or not this is the case there is no doubt that Autolycus was strongly influenced by the views of Eudoxus on astronomy. The above introduction in a natural way turns our attention to this field and the problem posing labs formed by different teams participating the 2009 math edition of the "UnSungHero" proposed problems from this field.

As a first step in spherical trigonometry some of teams proposed to study the following argument: the sum of angles of a curved triangle on a sphere.

Problem 1. Let A and B are points on the sphere with center O and radius R. We shall call "segment AB on the sphere" the arc connecting A and B and lying on the plane AOB. If A,B and C are three points on the same sphere and AB and AC are "segments on the sphere", then the "angle" A between them is defined by the angle between tangent lines to the arcs AB and AC. Find on the sphere a triangle ABC, such that the sum of "angles" A, B and C is 270°.

Despite of the fact this is an easy problem, this is not a well studied argument in the High Schools. Nevertheless, some of the solutions have been very interesting and original. Let's consider one of them.

Solution. (Solution proposed by the team ACUTANGOLI (Livorno))

This is a problem of spherical geometry, thus the Euclidean theorem of the sum of the internal angles of a triangle does not hold: as a matter of fact, the sum of the internal angles of a spherical triangle is directly proportional to the area of the triangle.

To simplify the calculations we consider R = 1. It is necessary to make it clear that with spherical wedge we have in mind : The portion of a sphere bounded by two maximal semicircles and with the poles of the wedge we have in mind the intersections between the two semicircles. By double spherical wedge we mean one wedge and his symmetrical wedge. The area of a spherical wedge S_F is directly proportional to the angle between the two semicircles, which can vary from 0 to π. We can set the proportion $S_F : S_s = \alpha : \pi$, with S_s area of the semicircle, in our case 2π. From where we obtain $S_F = 2\alpha$. For a double wedge we have

$$S_F = 4\alpha .$$

Any spherical triangle can be obtained by the intersection of three double spherical wedges. Actually, the intersection of three double spherical wedges generates two spherical triangles which are inversely congruent, in the figure are shown with thick line.

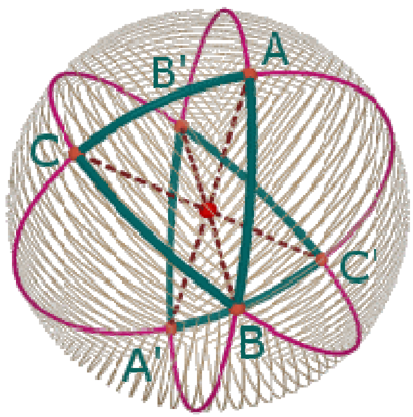

Figure 2. Spherical triangle as intersection of double spherical wedges.

We show the three double wedges from here above:

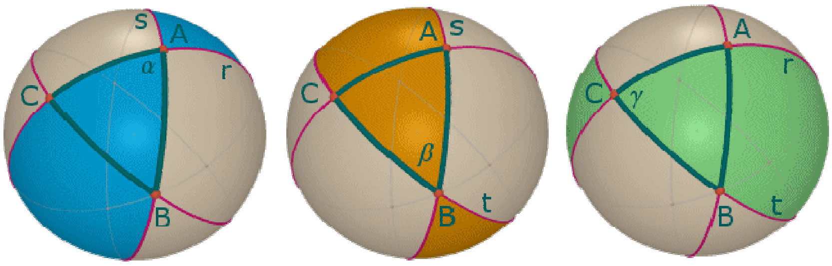

Figure 3. Three double wedges

We notice from the figures, that the three double wedges cover perfectly the entire sphere, but the triangles ABC and $A'B'C'$ are covered three times. The sum of the areas of the three wedges is equal to the surface area of the sphere, increased two times the area of ABC and two times the area of $A'B'C'$. Hence we can write

$$S = F_A + F_B + F_C - 2S_{ABC} - 2S_{A'B'C'}.$$

In our case $S = 4\pi$, $S_{ABC} = S_{A'B'C'} = S_T$, and because of the formula proved before $F_A = 4\alpha$, $F_B = 4\beta$ and $F_C = 4\gamma$. Substituting, we obtain:

$$4\pi = 4\alpha + 4\beta + 4\gamma - 4S_T.$$

Hence

$$S_T = \alpha + \beta + \gamma - \pi$$

It is required a triangle in which the sum of the internal angles is $270°$ or $\frac{3}{2}\pi$. We substitute and find $S_T = \frac{3}{2}\pi - \pi = \frac{\pi}{2}$. To have a sum of the internal angles of $270°$

it is necessary and sufficient that the surface area is $\frac{\pi}{2}$, or $\frac{1}{8}$ of the surface area of the sphere. One triangle that verifies that condition, which to simplify the explanation we will suppose drawn on the earth, is the one that has one vertex on the pole and the other two on the equator, on two perpendicular meridians.

Another example is one interesting product of the work in math labs, the pupils and teachers arrived at the following.

Problem 2. Let A and B are points on the sphere with center O and radius R. We shall call "segment AB on the sphere" the arc connecting A and B and lying on the plane AOB. If A,B and C are three points on the same sphere and AB and AC are "segments on the sphere", then the "angle" A between them is defined by the angle between tangent lines to the arcs AB and AC. Given triangle ABC on the same sphere, such that the "angle" A is 90°, say if the Pythagorean theorem is true i.e. AB^2+AC^2 = BC^2, where AB, AC and BC are the lengths of the corresponding "segments"? If the theorem is not true, then how one can modify the Pythagorean theorem.

This example shows that the problem -- posing activity can produce very nice and creative topics that are very close to the spirit of the math research.

The solutions found by different teams use different ideas, but our goal here is to stress the attention to such a possibility for attractive and fruitful work on beautiful math topics and even to ask the groups of math teachers and pupils in High School to create and pose new questions and problems.

We shall mention for completeness the answer to above problem:

$$\cos\frac{a}{R} = \cos\frac{b}{R}\cdot\cos\frac{c}{R},$$

where, *a=BC, b=AC, c=AB,* so after taking the limit as R goes to infinity we recover the Pythagorean theorem.

Let us now turn to the "innocent" problem proposed by the team of the High School "Dini" in Pisa.

Problem 3. (Problem "Moon Satellite") The satellite THETA of the society GoogleMoon is equipped with digital camera having image with a 20 degree angle of view. To make qualitative photos the satellite must circle around the Moon keeping constant altitude equal to the radius R=1738 km of the Moon and the digital camera must be oriented all the time in the direction of the centre of the Moon. Find the minimal length of the path of the satellite allowing to cover (with the images of the digital camera) the whole surface of the Moon.

To clarify and pose in more clear form the problem let us add the explanation of the team that proposed the problem. If the cone of the camera has image with a 20 degree angle of view, then the cone of the camera is a cone with angle of semiaperture 10 degree.

One can see that this problem is similar to the problem known by researchers in Pisa University. The open problem treated the square on the plane, while the problem of the team of "Dini" looks for the case of a sphere. Therefore, the real life problem lead to deep real math problem!

The difficulty of the problem posed the question how the teams could solve the problem, since they have no deep math tools, but even the specialists with heavy math preparation could not solve it. The team of Brescia proposed an interesting simulation using orange.

Feedback: Questions and answers

Questions have been answered by 27 participants in mathematical labs or in minicourse on the book "Bringing Mathematics to Earth".

Question 1: What do you learn from the book "MATH2EARTH" or from the group mathematical activity "Galilei" or from the course on "Math models in the real life and mathematical education"?

Answers:

1) (20%) I have learnt some new combinatorial argument from the book.

2) (32%)I never really thought that geometry has different rules and relations on a curved space.

3) (32%) I understood the mathematical models could be used in a real life situation.

4)(25%) I have learnt some new in solving optimization problems from the book.

Question 2) Explain the reason for your participation in the out-of-school activity "Galilei".

Answers:

1) (75%) Because I am motivated and have passion in participating math competitions.

2) (25%) My students convinced me to participate.

3) (55%) I was sure that the activity will increase the mathematical motivation of my students.

4)(45%) Since the rules are more flexible and different from standard mathematical (individual and team) competitions.

5) (10%) Because I have a friend, teacher in mathematics that convinced me to take part.

6) (45%) To win the award and travel with the ship that discovered America.

Question 3) Explain the part of the activity "Galilei" that was most attractive for you and your team.

Answers:

1) (45%) The first part and the problem posing work.

2) (35%) The Internet Competition.

3) (55%) Visit of Pisa.

4) (35%) Visit of “Amerigo Vespucci”.

5) (65%) To meet and know people from other countries.

Question 4) Explain the part of the activity "Galilei" that was most difficult for you and your team.

1) (75%)To solve the problem that remained unsolved even after the competition (see the problem "Moon Satellite" above).

2) (35%) Final competition in Pisa.

3) (15%) To find financial support to come to Pisa.

4) (10%) I met some language problems of understanding the problems posed at Internet competition.

5) (75%) Impossibility to find some free time for preparation of the team.

6)(12%) bureaucratical problems in our School to get a permission to participate.

Analysis and conclusions

In general the group approach applied in the project helped essentially to work with a larger group of students and teachers.

Compared with the individual mathematical competitions (for example the well - known individual mathematical Olympiads) it is not necessary to have a very well organized network covering the essential part of High Schools in a fixed country. To have successful national mathematical team at International Olympiads in mathematics it is necessary to make very restrictive selection and obtain a limited group of highly specialized pupils in solving concrete problems for limited time, therefore our project can enable one to avoid some of the weak points of individual approach to work with gifted students.

The possibility to work on concrete problem and even on mathematical project for longer period is an essential step to give a model of real mathematical research to pupils and teachers. The participating University researcher can help teachers and pupils to avoid the difficulties in posing and solving real life problems with mathematical tools. As it was explained in the introduction in some cases out-of-school problems are only apparently similar to school mathematics problems, but there are no simple algorithms or methods to solve the problem and school-learnt procedures are not sufficient.

The example of the "Moon Satellite" problem shows that such possibility can not be avoided completely, but in this case can eventually generate possible progress in developing new mathematical ideas and methods in the future.

Among other conclusions we can mention that some of the arguments on spherical trigonometry were not be studied in school at all. It is important to mention that the curvature of the sphere is constant and positive. Therefore, the Universe is bounded and positively curved. It is not clear how the situation will change if the curvature is negative? One can try to see possible examples. Obvious relations with astronomy and general relativity are manifested.

Possible tasks and further applications are listed below.

Problem 4. Suppose a town X has longitude 2°W, latitude 50°N. while the town Y has longitude 97°W, latitude 50°N. Find the distance between them ?

Problem 5. If you know the distance between X and Y , the longitude and latitude of X and only latitude of Y can you find the longitude of X?

Problem 6. Can you find a surface different from the sphere so that the sum of angles of a curvilinear triangle is less that 180°?

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VARYING VIEWS OF MATHEMATICS AND IMAGES OF MATHEMATICIANS AMONG NORWEGIAN UPPER SECONDARY SCHOOL STUDENTS

BARBRO GREVHOLM

ABSTRACT. Norwegian upper secondary pupils' varying views of mathematics and drawn images of mathematicians are explored. Earlier research on the image of mathematics is used as a theoretical foundation for the analysis of pupils' answers. The view of mathematics is mostly linked to numbers and calculations although there are more sophisticated views. The images of mathematicians show an old, lonely man, often a bit strange-looking and rather passive. It seems as if the profession of mathematician cannot be attractive for young students. The myth of the eternal, invariant mathematics is challenged by the individual views of young students.

Introduction

It is often claimed that mathematics is hidden in the modern society and mathematicians are a very silent group of professionals. As part of collaboration between school and university I was invited to an upper secondary school to meet students and talk about mathematics. The theme of the visit was "Meet an Agder- researcher". I was supposed to present myself and my work to them as a mathematician and mathematics educator. In that situation it was interesting for me to know what conceptions they already had of a mathematician. In a try to be systematic I took the opportunity to ask the pupils to reply to two tasks before I started my presentation. The two tasks were to write down your own thoughts on "What is mathematics?" and to draw a picture of a typical mathematician. They willingly did so (and afterwards via the teacher I got their acceptance to use the products in a paper). Their work gave me an opportunity to investigate what pupils in a normal class in Norway see as mathematics and how they imagine a typical mathematician. The pupils were specialising in design and media and this might have influenced their ability to express themselves in drawing in a positive way. The school was a normal upper secondary school in one of the larger cities in Norway and the class consisted of 12 pupils.

Related literature and theoretical framework

For an overview of research about "the mathematics image problem" I refer to Picker and Berry (2000). They sketch the development in research since the Draw-A-Scientist-Test initiated by Mead and Métraux (1957). Other researchers have approached the view of mathematics in different ways.

Pehkonen and Törner (2004) introduced a three component model for belief systems with a tool box aspect, a system aspect and a process aspect. A fourth component was added later as the role of applications within mathematics (Grigutsch, Raatz & Törner, 1998). These researchers built on work of Dionne (1984), who pointed to the three perspectives: traditional, formalist and constructivist. They represent mathematics seen as a set of skills, as logic and rigor or as a constructive process, respectively.

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Roberta Mura (1993) has investigated images of mathematics held by university teachers of mathematical sciences. The question "How would you define mathematics?" was posed in a questionnaire to 173 university teachers. The analysis led to identification of the following themes: 1) The study of formal axiomatic systems, of abstract structure and object, of their properties, and relationships; 2) Logic, rigour, accuracy, reasoning, especially deductive reasoning, the application of laws and rules; 3) a language, a set of notations and symbols; 4) design and analysis of models abstracted from reality, their application; 5) reduction of complexity to simplicity; 6) problem-solving; 7) the study of patters; 8) an art, a creative activity, a product of imagination, harmony and beauty; 9) a science, the mother, the queen, the core, a tool of other sciences; 10) truth; 11) reference to specific mathematical topics (number, quantity, shape, space, algebra etc). Mura concludes that the fact that some views, like formalism, are widespread among university teachers may explain and justify their prevalence among school teachers. Additionally changing school teachers views may be an ambitious project for it may involve contradicting ideas they have received during their training. Or it may involve changing university teachers' views as well (ibid, p 384).

A more recent way to present what characterises a mathematician is given in the competence model by Niss (2004). Eight competencies are seen to be the constituents in mathematical competence: the competency of 1) mathematical thinking, 2) problem handling, 3) modelling, 4) reasoning, 5) representation, 6) symbols and formalism, 7) communication and 8) tools and aids.

Picker and Berry (2000) have investigated pupils' images of mathematicians. Pupils aged 12-13 years from United States, United Kingdom, Sweden, Romania and Finland were asked to draw a picture of a working mathematician. Their examination for commonalities in the 476 pictures identified these sub themes: Mathematics as coercion, the foolish mathematician, the overwrought mathematician, the mathematician who cannot teach, the Einstein effect, and the mathematician with special powers. Picker and Berry propose a cycle of the perpetuation of stereotypical images of mathematicians and mathematics (see figure 1).

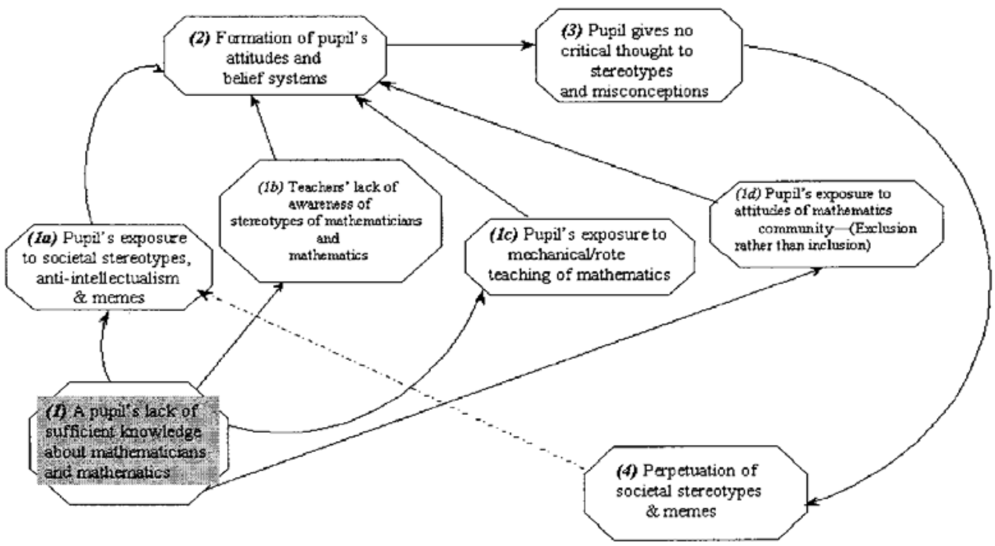

Figure 1 The proposed Picker – Berry cycle of the perpetuation of stereotypical images of mathematicians and mathematics (Picker & Berry, 2000, p 86)

Among implications for pedagogy and conclusions Picker and Berry (2000) mention that teachers appear to be largely unaware of pupils' lack of knowledge about mathematicians and the role teachers can play in shaping and changing pupils' views about them (p 89). "Teachers need to learn with greater clarity what it is that mathematicians do and there is no reason why they cannot do this alongside with their pupils." (ibid, p 90) Picker and Berry see the tool they have developed as an effective beginning for ascertaining pupils' beliefs about mathematics and mathematicians. It can be a means to challenge negative views and stereotypes.

Maria Bjerneby Häll (2002, 2006) followed a group of student teachers during their teacher education and the first three years after their debut as mathematics teachers in school. One aim of her study was to describe and analyse arguments for mathematics in compulsory school. During the education the student teachers develop a view on mathematics and mathematics education that is in harmony with the goals of mathematics in the national syllabus. The main arguments that were presented by the student teachers were to manage with ones everyday life, for future education and occupation, to take care of ones own interests in society, because society needs and demands this knowledge, to develop thinking skills, because it is funny and will increase ones self-confidence, it is needed in many other school-subjects, it is an important body of knowledge and a language, it is part of our culture and common knowledge, and because there will be a test.

The images here are analysed by picture analysis (Långström & Viklund, 2010) asking how persons and things are grouped/placed in the image, what age and gender they have, and how they are dressed and moving. Further I describe the background, foreground, the middle, left and right in the picture, if any. Validation was done by comparing with the descriptions made by a colleague.

The students in this study have an age between those in the Picker and Berry study and in the studies of student teachers and teachers. I will make use of theoretical elements from several of the above mentioned studies in the analysis below.

Views of mathematics

The pupils' written answers to the question "Hva er matematikk?" (What is mathematics?) will be presented here before the analysis. The exact wording in Norwegian is given first and then my translation in parenthesis afterwards. I have numbered the replies 1-12 in order to be able to link the respective drawings to the descriptions of mathematics.

1 Regning med tall (Calculations with numbers)

2 Læren/kunnskapen om tall, utregninger… (The theory/knowledge about numbers, calculations…)

3 Læren om tall og deres funsjon (The theory about numbers and their function)

4 Jeg tror matematikk er læren om tall og deres funksjoner. Matematikk er også en måte på å bruke hjernen til å utløse problemer. (I think mathematics is the theory about numbers and their functions. Mathematics is also a way to use the brain to solve problems).

5 Tall, formler, likninger, naturvitenskap. (Numbers, formulas, equations, natural science).

6 En ting du bruker hver dag, en måte å finne ut ting på med tall. (Something you use every day, a way to find out about things with numbers).

7 Matematikk er et verktøy for å løse små og store problemer. Med hjelp av matematikk kan vi lage forenklede modeller av virkeligheten og forutsi et svar på et problem.

(Mathematics is a tool for solving small and large problems. With the help of mathematics we can create simplified models of reality and hypothesise an answer to a problem).

8 Matematikk er tall og regning. (Mathematics is numbers and calculations).

9 Mattematikk er tall satt i systemer med system som kan multiplisere, dividere osv. Matte er i alt. Alt er matte. Stolene er matte, (Mathematics is numbers into systems with systems which can multiply, divide and so on. Maths is in everything. Everything is maths. The chairs are maths,

10 Matematikk forbinder jeg med tall og størrelser og forholdet mellom dem. Kanskje bruker vi matematikken til å kartlegge våre omgivelser. (I relate mathematics to numbers and quantities and the relation between them. Maybe we use mathematics to map our surroundings).

11 Tall eller symboler som legges sammen i en sum. (Numbers or symbols put together into a sum).

12 Ulike ”verdier” og samspillet mellom dem. (Different “values” and the interplay between them).

Images of mathematicians

The twelve images I got from the pupils are enclosed in the appendix. I will now try to describe them as fully as possible before I analyse them. The copies of the original images are in the appendix.

Drawing number one has a text above it:”The typical mathematician:”. The picture shows a nice looking man in suite with his hands folded together in front of him. The age is indefinite; he could be young or old. The face looks calm and kind and both eyes and mouth are large and visible. The mouth seems to have a small smile on it. The hair is curly and large but not un-normally large. He looks like a normal man but maybe a little bit shy?

Drawing number two shows an older man with round spectacles and a pen behind his left ear. He seems to be dressed in some kind of laboratory coat. He holds a pointer in his right hand and rests it on the floor. The left hand is in his pocket. In his upper pocket we can see a pencil and a ruler sticking up. His trousers are a little bit too long. He is almost bald and has a skewed smile on his face.

Drawing number three has one part of an egg-shaped head on two feet which is crossed over. The other part is a man in trousers and naked upper body. His hair is large and curly, the mouth shows the teeth, the eyes are just two empty circles and there seems to be a small moustache. He is walking with his arms hanging straight down.

Drawing number four shows a head of a man on an upper body which is just indicated. He is partly bald and has horn-rimmed spectacles. He has a little bit of dark hair on the sides of his head and he carries a black beard. The mouth is closed and expressionless. He is squint-eyed and speaks out a mathematical expression in a bubble. He looks a little bit frightening and cold.

Drawing number five shows a walking man with a rather small body and a large head with a long neck and the Adam’s apple visible. He has a beard and the mouth is open and shows big teeth. The nose is big and the spectacles even bigger but the eyes are just two black dots. On the top of his head there are two straws of hair standing up. He looks frightening somehow.

Drawing number six has an explaining text, which says: “I see an old man with a lot of hair on his head and spectacles sitting at a desk doing calculations on many pages.” My interpretation is that the man is leaning deeply over the pages on the desk and writing intensely. The bodily expression of the man is friendly.

Drawing number seven shows a man sitting at a desk, scratching his long thin black hair so that the scurf is whirling around. On the desk there is a cup of steaming coffee and a beaker with many stumps of cigarettes, some still giving smoke. The face is strange with two eyes on each side of what could be a very long nose and a big mouth dividing the face in two parts. The upper teeth are shown in a long straight row and there is growing beard on the cheek. The image has the label ‘The mathematician’.

Drawing number eight shows a man sitting at a desk in a room with a bookshelf full of books, a blackboard full of notes (12+4+x+a3=~ ….) and paper basket full of crumpled papers. On top of the desk there is a book in front of the man, a ruler, a computer (?) and a paper. The man is thinking √25 + mc2, which is shown in a bubble. The man has curly hair and spectacles and a little smile on his face. There is peacefulness in the room.

Drawing number nine is a head of a man, drawn in red ink. It is a triangle shaped head with big unordered hair and a long tapering beard. The mouth is dark and hidden in the beard, the nose is strong and the eyes are narrow. The face is somewhat expressionless.

Drawing number ten shows a man sitting in a dark room at a desk with a strong lamp enlightening his working area. He works on a computer. On the floor there is a striped home-woven carpet and two pictures on the wall. One picture shows sun, the other the number 4 (could indicate a calendar?). We see the man from the back so the face is not accessible.

Drawing number eleven shows a man walking and above his head is written 3 times √9 raised to 12 π s. He has short black hair and an expressionless face. The hands are drawn in a childish way with just five lines as fingers.

Drawing number twelve is the upper half of a man with spectacles and dark hair close to his head. The eyes are wide open and somehow starring. The nose is small. In his pocket is a pencil. Beside him is a cascade of mathematical expressions springing out of a hole (2+3=5, √4 divided by 22 times x b3=z, E= mc2). He has a lot of dots in his face and maybe a cigarette in his mouth?

Analysis

In the answers to what is mathematics the most common word is numbers (mentioned 10 times) and second common is calculations and to solve problems (mentioned 3 times each). Although the answers are rather short taken together they represent several of the categories found by Mura. Mathematics is seen as a formal system, models and application, symbols, a tool, and there are references to arithmetic and relations. Mentioned are the mathematical objects number, problem, equation, model, quantity, formula, relation, symbol and operations such as calculate, solve, sum, multiply, divide and model. The three aspects from Pehkonen and Törner (2004) are present: toolbox, system and process. Even applications in the form of models of reality and prognoses are mentioned. The answers seem to be neutral or positive in attitude (something you use every day, it is everywhere) compared to the negative aspects given by younger students in the Picker and Berry study. On the other hand it is rather simple descriptions of mathematics compared to the answers from the more mature student teachers or teachers in the studies by Bjerneby Häll and Mura.

What can we learn from the drawn images of mathematicians? They are all men and most of them are old. They are all alone. In three cases (6, 8 and 10) a kind of working environment is indicated. The workplace is a desk in an office. The working tools we see are paper, pen, pencil, ruler, pointer, computer, books, numbers and mathematical symbols. Six of them are wearing spectacles. Three of them are walking, four of them sitting at a desk and the rest just standing still. Three of them have a beard. In four cases we see mathematical expressions linked to the men. These expressions seem to be used to indicate the activity of mathematical thinking.

There are a number of stereotypes given in the images. The mathematician is a man, not a woman. He is old, he is lonely, has often spectacles and sometimes beard. The content of the mathematician's work is hidden. We only get to know that he is working at a desk, he is thinking on numbers and symbols, writing or computing. But what is he actually doing? Only one person could be teaching, the man with the pointer. One has a blackboard in his room which could indicate teaching or at least talking to someone else about the work. Compared to the eight competencies mathematicians need to have (Niss, 2004) we can trace little of them in the images.

The drawings in this study are from pupils about five years older than the ones in the study by Picker and Berry (2000). This seems to be an important difference as we do not find the violence or intimidation of pupils seen there. The mathematician as a teacher is much less common and the foolish person is also missing. The common features are the nerd, the strange-looking person and the lonely person. In the images of the younger pupils there are pupils present in the pictures, which is not the case among the upper secondary pupils' drawings. The latter might not see their mathematics teacher as a mathematician?

Is becoming a mathematician an exciting future for a young student in school? Can they see the inspiring, aesthetic, challenging aspects of mathematics mentioned in Mura's study? It is hard to see how the students could identify with the images they have drawn. Where is the excitement of solving mathematical problems, where is the excitement of working together with other people, of teaching young students, of modelling diverse situations in society and assisting with solutions to important problems for the future? Such aspects of the work of a mathematician are obviously hidden to these students. They see the mathematician as someone very different from themselves.

Conclusions

Data in this paper consists of descriptions and drawings from only 12 pupils in upper secondary school, thus no general conclusions should be made based on it. But it raises a number of questions that might be further explored. Why are the pupils holding these views of mathematics and images of mathematicians? Would we like pupils to have other views and images? How did the pupils build up these views and images? Is it possible to break the "evil" cycle modelled by Picker and Berry? Do mathematics teachers ever discuss with their pupils about what mathematics actually is and what a mathematician is doing? Are the competencies of mathematicians (Niss, 2004) ever visible for students? The popular claim or myth that mathematics is stable and not changing, invariant, does not seem to fit either with the view mathematicians have of mathematics or the view students have of mathematicians. Students' views seem to vary although they might not be what mathematicians themselves could hope. We know little about answers to these questions. Lately some films and TV-series have been produced with mathematicians acting. Such examples could of course help inform pupils about the work of mathematicians. Even in those films of TV-series the mathematicians are quite often pictured as nerds or strange persons. We do not know if pupils see those films or if they get any impression at all from them. And it is very rare that working mathematicians want to take time to contribute to the public picture of mathematics and mathematicians.

For young persons today it is probably crucial to get another image of mathematics before they even consider studying mathematics or becoming a mathematician or scientist. Society and the educational system have a huge task here in making mathematics more visible to pupils in school.

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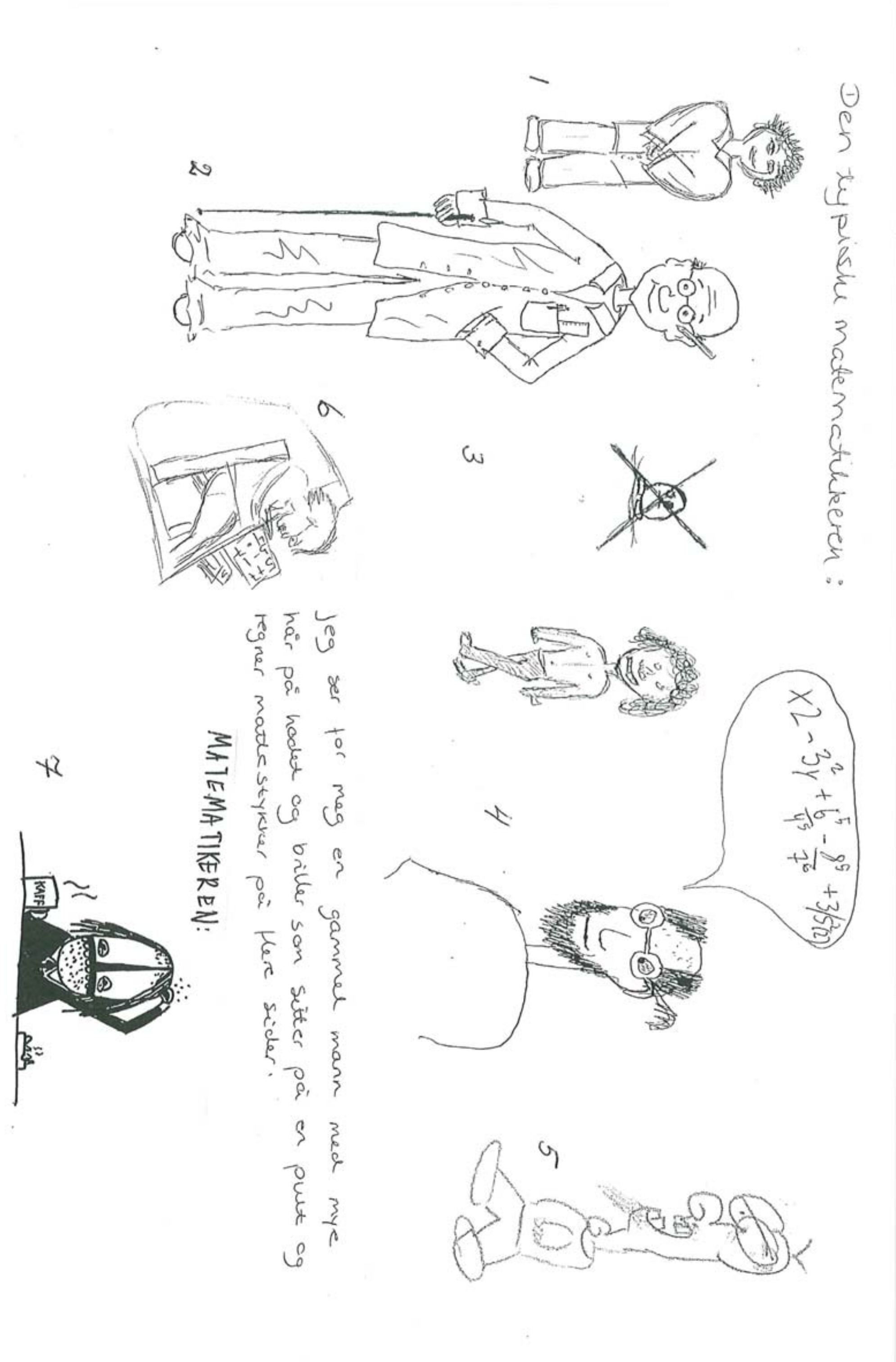
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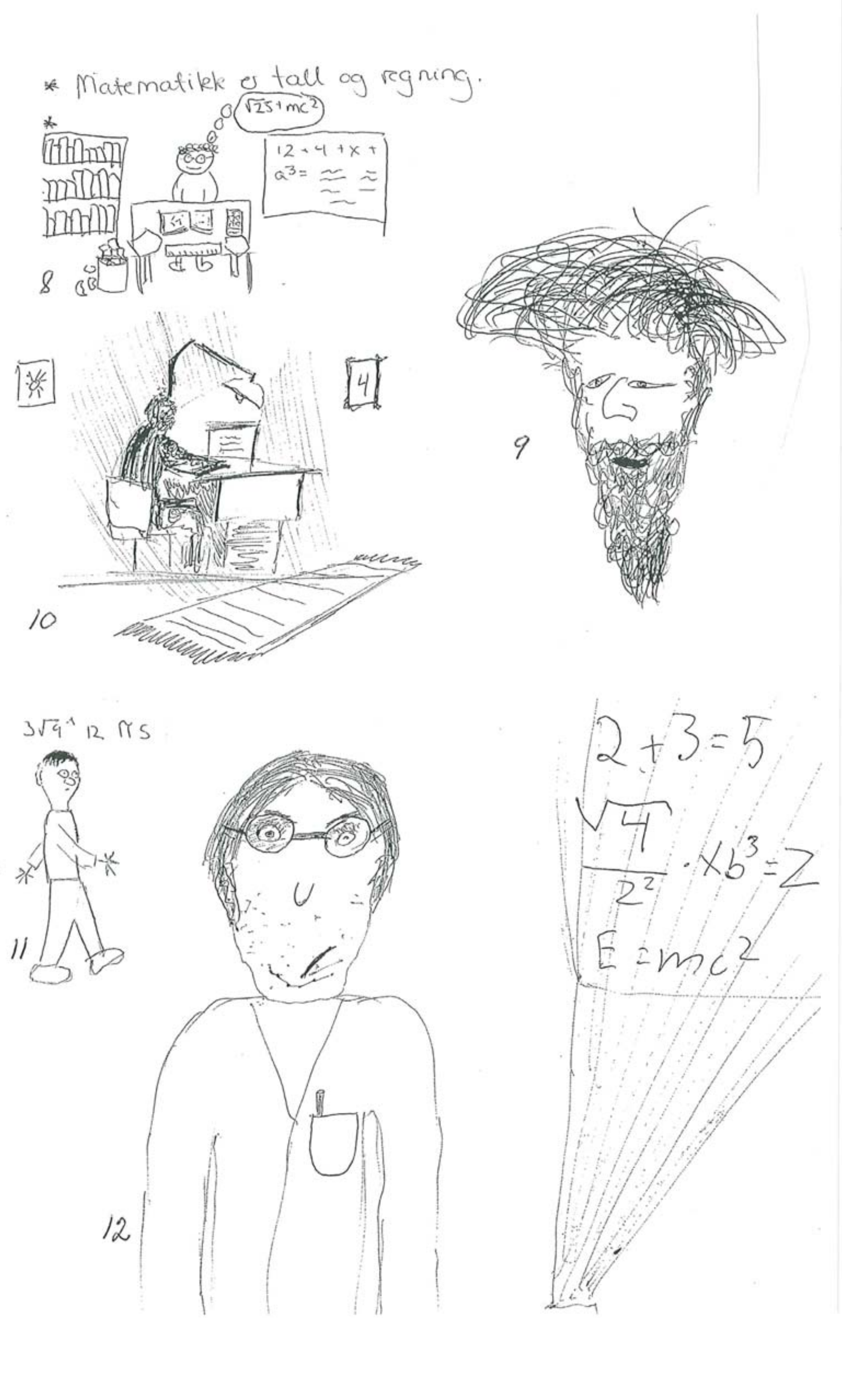
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MATHEMATICS AT WORK

HENK VAN DER KOOIJ

ABSTRACT. Official documents in many countries state that the goals of mathematics education are threefold: to prepare for *citizenship*, for *work* and for *further learning*. But in many educational practices, the main emphasis is still on procedural fluency, especially in algebra, needed for further learning. This paper describes how mathematics for and in work differs from mathematics learned at school and tries to seek for a better balanced mathematics curriculum in secondary education.

One mathematics, different settings

There are many research reports, based on empirical studies and other ones on more theoretical considerations, that discuss how mathematics embedded in work settings is different from the way mathematics is learned and practised in the typical traditional school settings (Hoyles & Noss, 1998; Forman & Steen, 2000). Different names, like school maths, work maths, street maths and other ones, seem to point to the fact that there are many different types of mathematics. However, I prefer to think of just one mathematics, with different approaches to it, depending on the goals for learning in a given educational setting.

As discussed in the PISA framework (OECD, 2002), a modern view on mathematics as a discipline emphasizes 'the study of pattern' and 'dealing with data' (Steen, 1990; Devlin, 1994). This is not a traditional content-driven look at a possible mathematics curriculum for school, but rather a way to describe the field of interest. The choice in the PISA study to describe the scope of mathematics education by four overarching ideas: (patterns in) quantity, (patterns in) shape and space, (patterns in) change and relationships and (dealing with) uncertainty opens up a novel way of thinking about a better balanced curriculum for (secondary) education that truly takes into account the threefold goal (society, work and further learning).

Characteristics of mathematics at work

> Mathematics in the workplace makes sophisticated use of elementary mathematics rather than, as in the classroom, elementary use of sophisticated mathematics. (Steen, 2003)

This phrase summarizes briefly the findings of many research studies on mathematics at work and other out-of-school settings. Where mathematics learned at school is embedded in a well-defined formal structure, the mathematics used in the workplace is embedded in the context of work. Practitioners at work do use situated abstraction in which local mathematical models and ideas are used that are only partly valid in a different context because they are connected to anchors within the context of the problem itself (Noss and Hoyles, 1996; Hoyles, Noss, Kent & Bakker, 2010).

The idea of situated abstraction is very helpful for understanding what happened in a reform project for senior high school vocational education (engineering) in the Netherlands

The project PRIMAS has received funding from the European Union, Seventh Framework Programme (FP7/2007-2013) under grant agreement n° 244380.

(van der Kooij, 2001). The principle was that mathematics should become supportive for the vocational courses and therefore the mathematical subjects were presented in the context of engineering. Although the students were considered to be low achievers in formal algebra, it was found that they could do algebra as long as the variables were real quantities with a meaning (time, length, speed, mass) – but that most of them got lost as soon as variables became abstract (x and y). An example: most students were able to do calculations on the pendulum equation $T = 2\pi\sqrt{\frac{l}{g}}$ with T (in s) the period, l (in m) the length of the pendulum and g (in m/s^2), but many of them had no idea how to deal with a more general (and more simple, but meaningless) equation like $y = 2\sqrt{x}$. For that reason, instead of going for full abstraction and generalization, we emphasized transfer from one context to another: how is a strategy or procedure similar in a different context and what is different. Most of the time, this transfer is not complete in the way that every context gives rise to its own modification of the method that is 'applied' in that context (Evans, 1999).

Some important general aspects of mathematics in context (of work) are (Bakker et al, 2008; Steen, 2001; Hoyles et al, 2002):

- reading and interpreting tables, charts and graphs,
- use of IT (like spreadsheets),
- dealing with numbers, often not precise and with units of measurement,
- proportional reasoning,
- statistical process control activities,
- representing and analysing data,
- multi-step problem solving.

Strange enough, most of these aspects are not found in mathematics curricula in secondary education.

Competences and skills for the workplace; the economic setting

Policymakers in the United States (SCANS, 1991) and Europe (European Communities, 2007) have described the competencies that are needed for future workers in a world in which "the globalization of commerce and industry and the explosive growth of technology on the job" (SCANS report) asks for skills that are different from the traditional ones learned at school. Competencies for work are described in the SCANS report (Steen, 2003) as the ability of people to use

- resources (allocating time, money, material, and human resources),
- information (acquiring, evaluating, organizing, maintaining, interpreting, communicating, and processing),
- systems (understanding, monitoring, improving, and designing),
- technology (selecting, applying, maintaining, and troubleshooting).

Skills needed for such competences are split up as follows:

- 'basic' skills: arithmetic, estimation, reading graphs and charts, logical thinking, understanding chance
- 'thinking' skills: evaluating alternatives, making decisions, solving problems, reasoning, organizing, planning
- personal qualities: responsibility, self-esteem.

The European Union describes seven key competencies for life long learning needed for personal life and for work. Key competence 3 (EU, 2007):

> Mathematical competence is the ability to develop and apply mathematical thinking in order to solve a range of problems in everyday situations. Building on a sound mastery of numeracy, the emphasis is on process and activity, as well as knowledge. Mathematical competence involves, to different degrees, the ability and willingness to use mathematical modes of thought (logical and spatial thinking) and presentation (formulas, models, constructs, graphs, charts).

Both the described aspects of maths in work and focus on the competences and skills described by the policymakers are hardly found in mathematics curricula for primary and secondary education.

Nevertheless, the intended change to competency based education (CBE, advocated in the European Union as the driving force for Life Long Learning) offers opportunities to reconsider the kind of math education that truly aims at goals related to the three settings: society, work and further learning.

Reconsidering mathematics at school?

If the threefold goals of mathematics education (citizenship, work and further learning) are taken seriously, the claims above should be considered when rethinking a balanced mathematics curriculum to achieve these goals. For vocational training this seems more straightforward than for general education. In vocational education the mathematical content should be defined in terms of applicability for the workplace setting. For engineering, some important key aspects are: direct and inverse proportionality, absolute and relative numbers, relationships between more than two quantities with dimensions and units of measurement included, reading and interpreting complex graphs, logarithmic scaling, tolerances and significance and a basic idea of chance in the context of uncertainty.

For general education, the PISA framework has the potential in helping to define a balanced curriculum. The four overarching ideas (Quantity, Space and shape, Change and relationships, Uncertainty) and the eight competencies (Thinking and reasoning, Argumentation, Communication, Modeling, Problem posing and solving, Representation, Using symbolic, formal and technical language and operations, Use of aids and tools) can be used to structure and design a balanced curriculum in which all three goals are met.

One example of a balanced curriculum in general education can be found in the Netherlands: Mathematics A (de Lange, 1987), designed for a special group of high school students: those preparing for a study in social sciences. It was a well-balanced whole of statistics and chance, discrete mathematics and applied calculus with many applications to and starting points within the context of several disciplines. Problem posing and solving activities, mathematizing and developing own strategies were seen at least as important as training for procedural fluency.

Discussion

> We believe, after examining the findings of cognitive science, that the most effective way of learning skills is "in context," placing learning objectives within a real

environment rather than insisting that students first learn in the abstract what they will be expected to apply. (SCANS, 1991)

Mathematics education that starts in (real life) contexts can have, if designed well, a natural flow from concrete to general/abstract. As stated before, a sense-making learning line does not need to end in the formal, abstract world of pure mathematics for all students. For most students, especially those who don't aim at a college level study in natural sciences, situated abstraction with transfer from one context to another is motivating and enough to get prepared for work and for living as a "constructive, concerned and reflective citizen" (OECD, 2002).

So, why are curricula in general not balanced? Maybe because of the deeply rooted belief that mathematics has to be learned in a linear way - from arithmetic, via algebra to functions (linear, quadratic, polynomials, exponential and periodic) and calculus, with some geometry and maybe some statistics alongside. This is how most curricula are designed and it does not reflect the modern view on the nature of mathematics (Steen, 1990; Devlin, 1994).

One of the benefits of studying mathematics in the workplace is that we can see that the mathematics used in different workplace settings and in daily life is far from 'standard' – and it is learned in these settings in different ways, many of which are far from 'linear'.

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STUDENT DEVELOPMENTS IN MODELING REAL WORLD PROBLEMS USING CONCEPTUAL TECHNOLOGICAL TOOLS

NICHOLAS G. MOUSOULIDES

ABSTRACT. This paper argues for a future oriented approach to mathematical problem solving in elementary and secondary school, one that draws upon the models and modeling perspective and technological tools that build on and extend students' conceptual understandings. Results revealed that modeling as a means for introducing real world problems to young students in conjunction with conceptual technological tools can enrich student explorations and understandings in mathematics.

Theoretical Framework

Students nowadays face a demanding knowledge-based economy and workplace, in which they need to deal effectively with complex, dynamic and powerful systems of information and be adept with technological tools (Lesh & Zawojewski, 2007). This situation places a strong emphasis on developing students' abilities to successfully use technological tools in dealing with complex problem solving for success beyond school. The latter is also underlined by a number of educationalists and organizations (National Research Council (NRC), 2001; National Council of Teachers of Mathematics (NCTM, 2000). An appropriate medium for achieving this goal for students is mathematical modeling, a process that describes real-world situations in mathematical terms in order to gain additional understanding or predict the behaviour of these situations (Lesh & Doerr, 2003; Mousoulides & English, 2008). Using a modeling perspective, students have opportunities to create, apply and adopt mathematical and scientific models in interpreting, explaining and predicting the behaviour of real-world problems.

In this paper models are considered as "systems of elements, operations, relationships, and rules that can be used to describe, explain, or predict the behaviour of some other familiar system" (Doerr & English, 2003, p.112). Further, we adopt the perspective of modelling as a cyclic process which includes the following steps: (a) Understand and simplify the problem. This included understanding text, diagrams, formulas or tabular information and drawing inferences from them. (b) Manipulate the problem and develop a mathematical model. These processes included identifying the variables and their relationships in the problem; constructing hypotheses; use strategies and heuristics to mathematically elaborate on the developed model. (c) Interpreting the problem solution. This included making decisions and analysing a system or designing a system to meet certain goals. (d) Verify, validate and reflect the problem solution. This included constructing and applying different modes of representations to the solution of the problem; generalize and communicate solutions; evaluating solutions from different perspectives; critically check and reflect on solutions and generally question the model (Mousoulides, 2007, Mousoulides et al., 2009).

The activities presented are developed under ModelMath project, funded by Cyprus Research Foundation under Grant Humanities/Education/0308/12). The views expressed are those of the authors and do not necessarily reflect the views of the CRF.

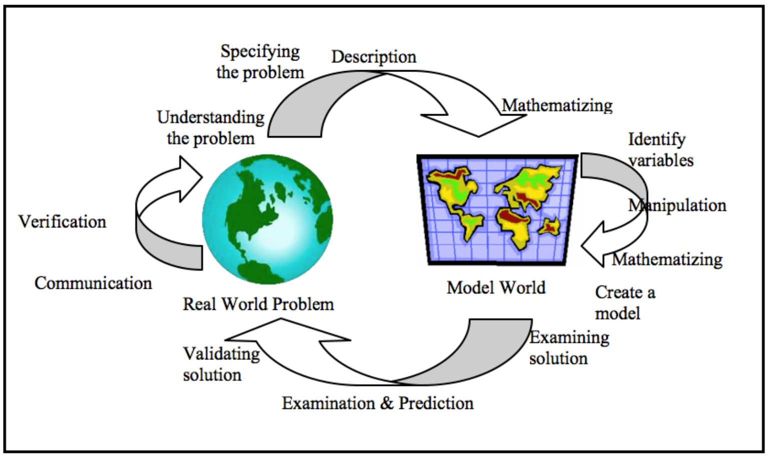


Figure 1: The Modeling Processes

In adopting the models and modeling approach, real-world based situations are presented to students. In modeling activities students are presented with complex real world problems that involve model development, and in which students repeatedly express, test, and refine or revise their current ways of thinking as they endeavour to create models that provide significant solutions - solutions that comprise core ideas and processes that can be used in structurally similar problems (Lesh & Doerr, 2003). Modeling problems engage students in mathematical thinking that extends beyond the traditional curriculum (Mousoulides & English, 2008). In contrast to typical classroom mathematics problems that present the key mathematical ideas "up front" and students select an appropriate solution strategy to produce a single, usually brief, response, modeling problems embed the important mathematical constructs and relationships within the problem context and students elicit these as they work the problem. The problems necessitate the use of important, yet underrepresented, mathematical processes such as constructing, describing, explaining, predicting, and representing, together with quantifying, coordinating, and organizing data (Mousoulides, 2007). Furthermore, the problems may allow for various approaches to solution and can be solved at different levels of sophistication, enabling all students to have access to the important mathematical content (Doerr & English, 2003; Mousoulides, Christou, & Sriraman, 2008).

In this paper we examine the impact of conceptual technological tools in students' explorations and model development. Recent studies showed that appropriate technological learning environments can have a positive impact on students' mathematical understandings in working with modeling activities (Lesh et al., 2007; Mousoulides et al., 2008). Further, these studies showed that the use of appropriate tools can enhance students' work and therefore result in better models and solutions. In Blomhøj's (1993) research, a group of 14 year old students were engaged in modeling activities with a specially designed spreadsheet. He reported that students did not find the spreadsheet was a barrier when they were setting up a model. Instead, they often expressed a given relation between variables in the model more easily in spreadsheet notation than in words. In line with previous findings, Christou, Mousoulides, Pittalis and Pitta-Pantazi (2004) reported that students, using a dynamic geometry package, modelled and mathematized a real world problem, and utilized the dragging features of the software for verifying and documenting their results. Similarly, Mousoulides and colleagues (2007) reported that students' work with a spatial geometry software broadened students' explorations and visualization skills through the process of constructing visual images and these explorations assisted students in reaching models and solutions that they could not probably do without using the software. In concluding, authors reported that the inclusion of appropriate software in modeling activities could provide a pathway in understanding how students approach a real world mathematical task and how their conceptual understanding develops.

This study builds on, and extends previous research by examining how a learning environment consisted of technological tools including dynamic geometry, spatial reasoning, and algebraic thinking tools can provide a pathway in better understanding how students approach and solve real world problems.

The Present Study

The results presented here are retrieved from two studies with one class of fourteen-year-old students and one class of twelve-year-old students used GeoGebra and Elica-Dalest in solving two modelling problems. The first modeling activity, *Solar Power Car*, requires students to develop quadratic function models for finding the best selling price for a solar powered car. The problem assigned to students was the following:

Problem 1: A car making company is launching a new solar powered car. A recent market research showed that one hundred people would buy the car for a selling price of 5000 euro. Further, the market research showed that for every 100-euro price increase, people interest in buying the car would decrease by one person. Find the best selling price for the car, as to maximize company's profits. Send a letter to the company sales manager, explaining how you solved the problem.

The second modelling activity, *Soft Drink Bottle Design*, requires students to generate by revolution 3D objects by moving and rotating a simple 2D object and to explore the properties of the constructed 3D objects, in order to design an appropriate bottle for a popular soft drink. The second problem that was assigned to students was:

Problem 2: Your group decided to participate in a competition for designing the best bottle for MarNiko® soft drink. If you want, you can use Elica-Dalest application for designing different bottles. After designing the bottle of your choice, write a letter, explaining and documenting your results, to the president of the competition. Try to convince her that your bottle is the best they could have!

Participants and Procedures

One class of 21 fourteen year olds and one class of 23 twelve year olds and their mathematics teachers worked on the two modeling problems. The students had only met such modeling problems before during their participation in a two year project on mathematical modeling, as the mathematics curriculum in Cyprus rarely includes any modeling activities. Students were quite familiar with using GeoGebra and Elica-Dalest, since the software was frequently used in the aforementioned project.

The data reported here are drawn from the two modeling problems, which entail: (a) a warm-up task comprising a story or an article, designed to familiarize the students with the context of the modeling activity, (b) "readiness" questions to be answered about the article, and (c) the problem to be solved. Working in groups of three to four, students spent four 40-minute sessions on each modeling activity. During the first session the students worked on the newspaper article and the readiness questions. In the next three sessions the students developed their models, and wrote letters to the company's sales manager, explaining and documenting their models/solutions, and presented their work to the class for questioning and constructive feedback. A class discussion followed that focused on the key mathematical ideas and the GeoGebra and Dalest constructions students had generated.

Software

GeoGebra (www.geogebra.org) is a Dynamic Mathematics Software (DMS) for teaching and learning mathematics from elementary school through college level. It is as easy to use as Dynamic Geometry Software (DGS) but also provides basic features of Computer Algebra Systems (CAS) and spreadsheet capabilities to bridge gaps between geometry, algebra and calculus (Figure 2).

Elica-Dalest (www.elica.net/site/museum/Dalest/dalest.html) consists of a number of applications for the teaching and learning of spatial geometry and transformations. The application used in this study was Potter’s Wheel. The main idea in the *Potter's Wheel* application is that the students have a simple 2D transformable segment passing through two to seven given points. As these points are dragged, the volume and surface of the rotational solids can be calculated along with the display of the “algorithm” (Figure 3).

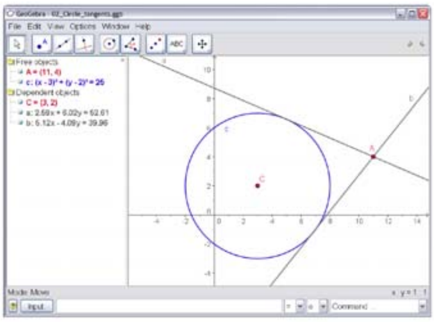


Figure 2: The GeoGebra Environment

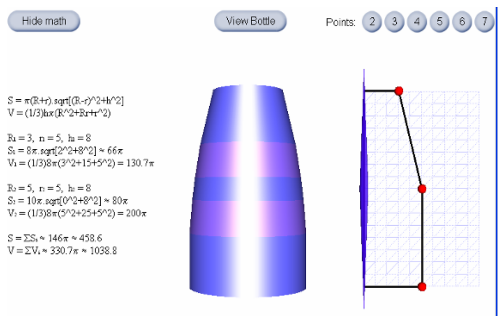


Figure 3: The Potter’s Wheel Environment

Data Sources and Analysis

The data sources were collected through audio- and video-tapes of the students’ responses to the modeling activity, together with the GeoGebra and Dalest files, student worksheets and researchers’ field notes. Data were analysed using interpretative techniques (Miles & Huberman, 1994) to identify developments in the model creations with respect to the ways in which the students: (a) interpreted and understood the problem, and (b) used and interacted with the software capabilities and features in solving the engineering problems. In the next section we summarize the model creations of the student groups in solving the *Solar Power Car* and *Bottle Design* modeling problems.

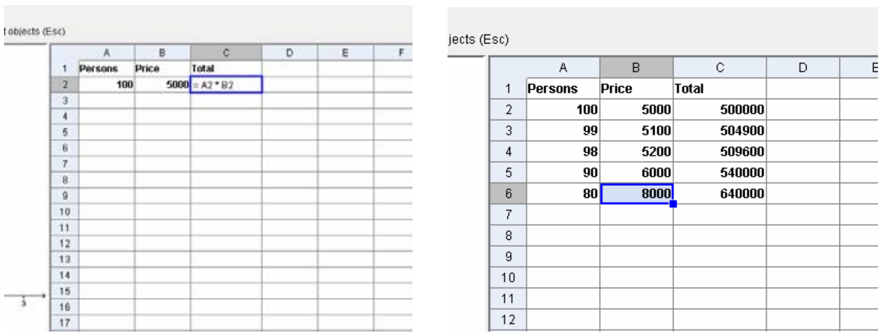
Results

Solar Power Car

The *Solar Power Car* problem required students to find the optimal price for selling a solar power car. Three groups of students successfully developed models using GeoGebra for solving the problem and their models are presented next.

Solar Car Model A

The first group commenced the question for finding the optimal price for the solar power car by brainstorming different factors that could have an impact on defining a car's price. Students also had long discussions about how they should start working on the problem and how they could use the software. One of the students pointed out that the price of an individual car is not that important, but the total amount of money the company will get is rather more important. This remark assisted students in understanding that they should find a method to calculate the total amount of money the car company would get from selling the solar power car. At this point, students moved into their GeoGebra construction, calculated (using spreadsheet's functions) the total amount of money and inserted the new data into the available spreadsheet (see Figure 4).



	A	B	C
1	Persons	Price	Total
2	100	5000	= A2 * B2

	A	B	C
1	Persons	Price	Total
2	100	5000	500000
3	99	5100	504900
4	98	5200	509600
5	90	6000	540000
6	80	8000	640000

Figure 4. Group A students' spreadsheet

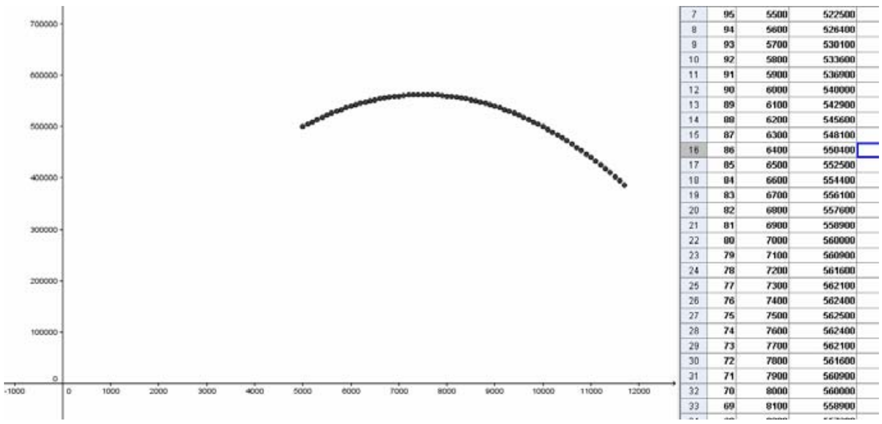
Students quite easily used the formula *persons*price* (see Figure 4) for calculating the total amount of money. However, students did not use any formula for calculating a new price and the number of persons that would be interested in buying the car. They "manually" enter the new data, instead of subtracting one person for each 100-euro increase, and as a consequence, they faced some difficulties in performing their calculations. Students finally succeeded in making the correct and necessary calculations and reached the optimal price for selling the car. In short, students took advantage of the spreadsheet capabilities of the GeoGebra, but they did not use GeoGebra as a dynamic geometry software for refining or extending their solution.

Solar Car Model B

Similar to the work presented in Solar Car Model A, students in this group reported that the question of the problem could be rephrased into finding the maximum profit the company would get. For the students in this group this decision was not, however, straightforward. On the contrary, students had long debates in understanding the core question of the problem. One student, for instance, reported that the problem was too easy; "the optimal price for selling a car is the starting price, since any change will have a negative impact on buyers' interest". As soon as they all agreed that €5000 (starting price) was not the best possible (by performing some calculations for increased prices), they made use of the software's spreadsheet capabilities in a similar way to group A; they calculated the total amount of money by multiplying the car price by the number of people.

Although students algebraically reached a solution to the problem, by finding that the optimal price was €7500, they also plotted a list of points, in an attempt to generalize their results and to further explore the relation between car price and company's profits.

Students listed the points defined by car price and total amount of money (see Figure 5). Although students did not face any difficulties in creating a list and its associated graph, they failed to explicitly discuss their findings; they enjoyed the fact that the software could handle both an algebraic (spreadsheet) and a graphical (geometry) representation of their solution. Students reported that the graph showed not only the optimal price, but also proved (according to their reports) that no other price could be better, since (7500, 562500) was the maximum point of the curve. In concluding, this group successfully used spreadsheet capabilities for finding the total amount of money, for calculating new price and new number of persons interested in buying the car. However, in plotting the points and in commenting on the shape of the graph, students failed to explicitly identify and discuss the graphs' shape and then to document why the specific car price was the optimal one. They also failed to document how their approach could be used in solving similar problems.



7	95	5500	522500
8	94	5600	526400
9	93	5700	530100
10	92	5800	533600
11	91	5900	536900
12	90	6000	540000
13	89	6100	542900
14	88	6200	545600
15	87	6300	548100
16	86	6400	550400
17	85	6500	552500
18	84	6600	554400
19	83	6700	556100
20	82	6800	557600
21	81	6900	558900
22	80	7000	560000
23	79	7100	560900
24	78	7200	561600
25	77	7300	562100
26	76	7400	562400
27	75	7500	562500
28	74	7600	562400
29	73	7700	562100
30	72	7800	561600
31	71	7900	560900
32	70	8000	560000
33	69	8100	558900

Figure 5. Student calculations and points plotting

Solar Car Model C

The model presented by the third group was far more sophisticated than Model A and Model B. Initially, students used formulae like $f(x)=ax^2$, $a > 0$. These first unsuccessful attempts helped them realized that the parameter a in the requested function should be $a < 0$, and that more parameters were needed. Further, since they were used to working with small numbers, they started their next round of explorations, using small numbers for parameters b and c. Students ended by formulating a general function $f(x)=ax^2 + bx + c$. The fact that students could not make even rough estimations of the parameters' values, encouraged students to use a simple yet very powerful tool of GeoGebra, sliders. The group C students' final workbook screen is presented in Figure 6.

Students spent a lot of time discussing, exploring and using trial and error to reach the correct function that represents the relation between car price and total amount of money. It was obvious that the software's features and capabilities helped them in performing actions, setting hypotheses, and investigating relations that they could not do with paper and pencil. During the last discussion with the researcher, students asked whether there was an easier way for finding the quadratic function, since they already knew (among other points) the coordinates of the vertex. The researcher encouraged them to search the Web and students easily found that the function could also be of the form $g(x)=a(x-h)^2 + k$, where h and k the coordinates of the vertex (in this example, h = 7500 and k = 562500).

Students quite easily constructed the graph of the function *g(x)* and they observed that the function representing the relation between car price and company's profits could be represented in the forms of $f(x)=ax^2 + bx + c$ and $g(x)=a(x-h)^2 + k$.

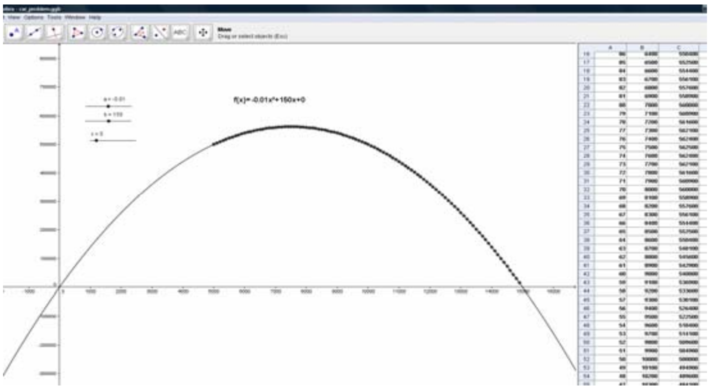


Figure 6. Function plotting and points

Bottle Design

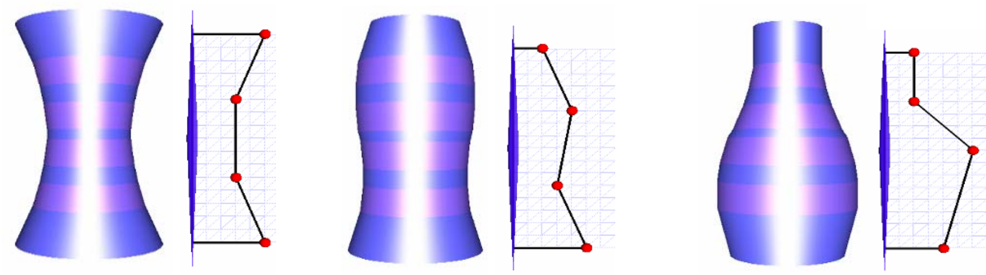
The results of the *Bottle Design* modelling activity are presented with regard to the two phases that arose from one group of three students' work. Students first drew their attempts in designing bottles with no attention to the mathematics part and during the second phase they made modifications in their designs, taking into consideration the surface and volume of the constructed bottle.

Focusing on non mathematical factors

Since in software's initial screen the segment is defined by two points, students' first attempts limited in constructing cylinders and frustum cones. Students commenced that most soft drink cans have cylindrical shape but these shapes are not nice and attractive. During their work, they did not pay any attention to the surface and volume measures. Students' next step was to select the three point segment. After a few explorations, they decided that it would be better to select a four point segment for designing more attractive bottles. The first bottle designed was a cylindrical one. One of the students commented that this bottle is the one used by most of the aluminium cans and it is too common for winning the competition. The same student suggested moving the two middle points closer to the axis and the bottom point to the opposite direction. The new bottle was more attractive. According to students' words, its "rocket" like shape was nice, but not convenient: "How can you hold it?" a girl commented. Students' next experimentations resulted in a number of "sine and curve shaped" bottles. Students commenced that these bottles are practical and attractive, since someone can both hold them easily, directly drink from them and there is enough surface on them for the company's logo. Quite interestingly, students performed a huge number of experimentations. These experimentations were not random. Students moved one point at a time, observing the result in the constructed bottle. After a few experimentations, the software helped them to visualize the expected result from each point movement in the bottle construction!

Beginning mathematization

After reaching a number of possible bottles, a discussion followed. During this discussion, the researchers prompted students to firstly decide the volume of the bottle and then modify the design to fit the volume of the initial cylinder which was 400π cubic units. They decided to work with the three designs they reached earlier, which are illustrated in their final format in Figure 7. They modified their designs to fit the volume, writing in their worksheets bottle's properties. Their initial attempts focused on moving points and observing the changes in bottle's properties. Quite interestingly, at a latter stage, they were hypothesizing what would happen in bottle's properties and more specifically in bottle's surface and volume. When they reached their final three bottles, they wrote in their worksheets the surface and volume measures for each bottle.



(a) (b) (c)

Figure 7. The constructed bottles and the 2D shapes

Since each student was supporting a different bottle, even after writing down the properties' measures, a discussion with researchers took place. During this discussion, researchers encouraged students to think about the different possible factors that the competition board would take into account and the importance of each factor. They all agreed that all bottles were attractive and practical. One of the students commented that the cost is the most important factor for all companies. This comment started a new round of discussion between students to find out the cheapest bottle in terms of aluminium or glass needed. The same student suggested the cost, pointed out that they should consider the surface of each bottle, since the volume was equal. Based on this suggestion, students agreed that the bottle in Figure 7c was the optimal one and the bottle that would cost the most was the one in Figure 7a.

Discussion

There are a number of aspects of the studies presented here that have particular significance for the use of modeling activities and technological tools in elementary and lower secondary school mathematics. Results showed that students were able to successfully work with mathematical modelling activities when presented as meaningful, real-world situations and when appropriate technological tools are available. When working on the problems presented in this chapter, students progressed through a number of modeling cycles, from focusing on subsets of information through to applying mathematical operations in dealing with the data sets, and finally, identifying some trends and relationships. In doing so, students successfully employed a number of software's tools and capabilities. There was evidence that GeoGebra's and Elica-Dalest's features and capabilities assisted students in modeling the real problems, and in making connections between the real world and the mathematical world. Further, the software assisted students in familiarizing themselves with the problem and in broadening their explorations and visualization skills through the process of constructing visual images to analyze the problem, taking into account their informal and visual conceptions.

An interesting aspect of this study lies in the students' engagement in self evaluation, through the use of software's features and tools: groups were constantly questioning the validity of their solutions, and wondering about the representativeness of their models. This helped them progress from focusing on partial data to generalizing their solutions and in identifying trends and relationships in creating better models. Although only a few students progressed to more advanced models in both problems, they nevertheless displayed surprising sophistication in their mathematical thinking. The students' developments took place in the absence of any formal instruction and without any direct input from the classroom teacher during the working of the problem. We can conclude that the conjunction of GeoGebra's environment and the framework of modeling activities enhanced students' models and solutions, and allowed researchers to gain insights into students' mathematical understandings.

Another conclusion of this chapter is that the younger participants were also able to work successfully with mathematical modelling activities when presented as meaningful, real world based problems. The learning environment based on the Elica-Dalest helped students to realize and to get familiar with the activity and thus enhanced their explorations and mathematical understandings. At the same time, the software broadened students' explorations and visualization skills through the process of constructing visual images to analyze the problem, taking into account their informal and visual conception. It was also clear that students identified the structural elements of the problem in developing their final model. The three factors taken into consideration (attractiveness, practical and cost) were employed by students in such a way that they could easily transfer and modify to create successful models for a structurally similar problem (Lesh & Doerr, 2003). The application engaged students in analyzing, planning and revising their actions to improve their solutions.

In preparing students for being successful mathematical problem solvers, both for school mathematics as well as beyond school, rich problem solving experiences starting from the elementary school and continuing to secondary school needs to be implemented and appropriate technological tools like GeoGebra and Elica-Dalest need to be effectively used in solving these real-world based problems. Results from research work like the studies presented here can provide both teachers and curriculum designers with details on how complex modeling activities with technology can assist students in accessing higher order mathematical understandings and processes. Further research towards the investigation of software's role is needed, in promoting students' conceptual understandings and mathematical developments in working with modeling activities.

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EXPLANATION IN TEACHING MATHEMATICS – TEACHERS' VIEWS

JARMILA NOVOTNÁ

ABSTRACT. This paper focuses on teachers' views on explanations in teaching and learning mathematics. The answers of the experimental group of mathematics teachers at secondary level are contrasted to the answers of two control groups: primary school teachers and secondary school teachers of non-mathematical subjects. Possible gaps between teachers' theoretical views and their everyday practice in school are investigated. Various types of explanations are characterised and examined with respect to their pedagogical, cognitive and social impact on learners. The theoretical frame for the research is based on the Theory of didactical situations (Brousseau, 1997).

1. Explanation

There is no doubt that explanation falls into everyday language. But what exactly is it? Is there a widely accepted definition of explanation?

(Longman Family Dictionary, 1991) states that explanation is "the act or process of explaining something", where to explain means "to make plain and understandable or to give the reason for or cause for". When consulting internet, e.g. the following definition of explanation can be found:

- "A set of statements constructed to describe a set of facts which clarifies the causes, context, and consequences of those facts." (en.wikipedia.org)
- "A detailed commentary aimed at illuminating a subject matter or its difficult parts in a manner that its underlying concepts and their linkages become clear and comprehensible." (www.businessdictionary.com/definition/explanation.html)
- "The process of making something intelligible or saying why certain things are as they are, or the account used to do these things." (http://www.bookrags.com)

There is a fundamental difference between explanation and arguments: "While arguments attempt to show that something is, will be, or should be the case, explanations try to show *why* or *how* something is or will be. … In this sense, arguments aim to contribute *knowledge*, whereas explanations aim to contribute *understanding*." (en.wikipedia.org)

In this paper explanation is characterized by its function as "a tool that is used by a speaker for understanding or 'giving a sense' to the object of communication, of a debate, or a discussion … The role of an explanation is to make clearer the meaning of an object (method, term, assignment) maintaining formally the necessary distance between the object of the action or study and the tools." (Mopondi, 1995, p. 12).

2. Explanation in teaching/learning process

Traditionally, explanation belongs to monological teaching methods where the information is transmitted in the direction teacher to students (together with e.g. narrative, description or lecture). Its goal is to manifest comprehension. (Skalková, 1999) states that

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in practice, individual forms of explanation often pervade. In this perspective, explanation is seen as the task fulfilled by the teacher with students passively receiving what is presented.

In the learning/teaching process we see explanation in a much broader sense as a tool used by both the teacher and the learners as a mutual interchange of information among participants of the educational process. The learners' role in the whole process is active (Mareš, Křivohlavý, 1995).

Using explanation in mathematics classroom is a common procedure, but its roles and forms vary. Predominantly explanation is seen as a tool for describing relevant phenomena, for development of students' logical thinking, and for guiding students to generalising by inductive judgement. It leads to clarification of interrelations, demonstration and justification (Skalková, 1999, p. 172). Although explanation is not often explicitly studied in literature, it has an important impact on communication in mathematics classrooms.

In (Novotná, 2005), explanation in teaching elementary mathematics (primary school level) was studied. In this contribution we focus on the secondary school level. Similarly to (Novotná, 2005), for classification of varieties of explanation we focus on the following variables (where [1] refers to the general situation, [2] the individuals involved, [3] to the academic/ intellectual context, [4] to the result, [5] refers to the formal structure of the explanation):

<u>Type of situation</u> (teaching/learning, communication, discussion) [1]

<u>Type of occurrence</u> of the explanation (necessary for the progress of a didactical action, arising naturally from the situation, accidental, etc.) [1]

<u>Transmitter</u> (author) of the explanation (teacher, student(s)) and <u>receiver</u> of the explanation (teacher, one student, group of students) (one-one; one-many; many-one) [2]

<u>Nature of the demand</u> for explanation (proposed by the transmitter, demanded by the receiver, generally spontaneous); linked with the <u>activity of students</u> (active – student produces an explanation during a discussion, passive – student receives an explanation as a piece of information) [3]

<u>Objects</u> of the explanation (explanation of an order, a mistake, a statement, an algorithm etc.); linked with the <u>purpose</u> of the explanation (for the teacher: to obtain information about students, e.g. to detect the location a misunderstanding/mistake, to eliminate a misunderstanding/mistake; for students: to build up knowledge, to find an appropriate solving procedure, etc.) [3]

<u>Results</u> of the use of explanation (for a teacher: facilitation of students' comprehension, feedback about the level of students' comprehension, etc.; for students: inclusion of a piece of knowledge into an existing knowledge structure, building a new structure, conversion of knowledge into a tool for problems solving, etc.) [4]

<u>Character</u> of the explanation (a range of levels from purely formal to a deep involvement in the explanation object) [5]

<u>Form</u> of the explanation (e.g. description, reformulation, visualisation, presentation of examples or counterexamples); linked closely with the <u>age</u> of participants [5]

Language of explaining (a range of levels from the exact language using special terminology to the natural language without any "scientific" precision) [5]

Frequency of the use of explanation [5]

The above proposed variables clearly show that the diversity of explanation in learning/teaching process is immense. This diversity leads to difficulties in detecting each of them in teaching/learning situations.

3. Explanations in teaching/learning situations in school mathematics

The paper focuses on secondary teachers' theoretical perception of explanation and their own school practice. It deals with two main questions:

Theoretical perspective: What modes of activity do mathematics teachers accept as explanations in their teaching (in cases of both types of transmitters)? What differences do they perceive and employ in relation to the age of the students?

Practical perspective: How do teachers project the theoretical perspective into their own teaching practice?

When collecting teachers' views, our main goal was to map the situation with mathematics teachers. In order to detect more clearly which of the views expressed are more general and which of them are specific for the teaching/learning mathematics, three groups of teachers were selected for participation in the investigation: Group A consisted of 17 secondary school mathematics teachers and two control groups – Group B consisted of 3 primary school teachers (to reinforce detection of age factors) and Group C of 7 secondary teachers of non-mathematical subjects (to reinforce detection of what is specific for teaching/learning mathematics). The comparison of the theoretical and practical part focused on detection of the gap between the theory and school reality.

Questionnaires (see Appendix) addressing both cases of transmitters – a) a teacher, b) a student were used and analysed. The questionnaires were divided into two main sections, each of which dealt with one of the group of questions: the "Theoretical part" with questions covering the forms and implementation of explanation; and the "Practical part" with questions reflecting teachers' own practice in both cases a) and b).

3.1. Analysis of teachers' responses to the questionnaire

3.1.1. Theoretical perspective

Forms of explanation

Question: What do you consider as an explanation? (Underline the types that you consider to be an explanation, possibly add further that are not listed.)

Description, rephrasing, definition, proof, presentation of a counterexample, commentary, illustration, use in another situation, analogy, decomposition into sub-cases, model.

Group A: The mostly underlined form was illustration (15/17[1]), the least selected was Definition (4/17). The frequency of choosing the presented types is summarised in Fig. 1.

Added forms: Demonstration of context, recalling previous experience

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[1] b/c expresses that b respondents from c choose the answer.

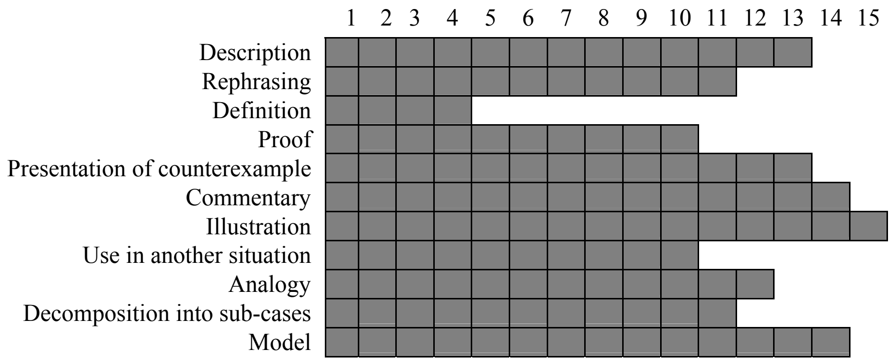

Fig. 1: Frequency of responses in Group A

Group B: All three respondents underlined Presentation of counterexample, Illustration and Model.

Definition was not underlined as an explanation suitable for primary level children. This can possibly be ascribed to the influence of the age of students.

Group C: All proposed types were underlined by at least 2 respondents. The items most frequently underlined were Description, Illustration, Use in another situation and Analogy, the least often underlined was Decomposition into sub-cases. It suggests that Decomposition into sub-cases is specific for mathematics (science) teaching. Three respondents (3/7) underlined Definition. This suggests that definitions in mathematics have a special status in comparison with non-mathematical domains.

Four questions related to two cases of the *transmitter* were posed:

(i) Which goals does explanation fulfil?
(ii) Which pitfalls does explanation bring?
(iii) In what phases of the teaching process do you consider explanation to be most effective?
(iv) How does explanation differ in relation to the age of students?

In the following text we present typical answers without stating their frequency. The answers are categorised according to who the answer speaks about: T – teacher, S – student(s). The answers that could be come across in more than one of Groups A, B and C are underlined.

a) ***Teacher as the transmitter***

Question (i), goals

Group A

T: Faster introducing of students into the subject matter (Group C), more effective
T: Higher number of possibilities to address students with different levels of mathematics abilities (Group C)
T: Presentation of interdisciplinary and intradisciplinary relations
T: Locating students' difficulties (Group C)
T: Delivery of knowledge to students with understanding
S: Illuminating concepts and processes that are not clear to students (Group C)
S: Help when solving problems

S: Achieving precision of concepts
S: Relating new knowledge to previous ones (Group C)
S + T: Mediation of contacts between the teacher and students (Group B, C)

Group B

In the answers, the possibility of deeper understanding of children's reactions which subsequently enables more appropriate action of the teacher (T) and understanding of relationships and conclusions (U) were emphasized. In comparison with Groups A and C, the answers were much more student than subject matter oriented.

Question (ii), Pitfalls

Group A

T: Extra work for the teacher – more preparation
T: Excessive teacher's use of monologue, transmissive way of teaching
T: Unsatisfactory focus on individual differences of students (Group C)
S: Unsatisfactory involvement of students in the teaching/learning process, increase of students' passivity (Group C)
S: Decrease of students attention (group B)
S: Decrease of students' responsibility for their education
S: Insufficient use of students' previous knowledge

Groups B and C

In these groups, the danger of using inappropriate language and insufficient understanding of the concepts used were emphasized. This was not the case in Group A. It seems that the reason for the difference is the influence of lower age in Group B and of the less exact nature of non-mathematical language used in other subjects on secondary school level.

Question (iii), Phases of teaching process

In all groups, answers are distributed into three domains: phases of the lesson – time, phases of the lesson – activity, (individual or group) needs. As no essential differences in the answers occurred, answers of all groups are summarised together. The answers do not look to be significantly age or subject matter dependent. The constructivist nature of the use of explanation was much emphasized.

Phases of the lesson - time:

- Most answers concerning this domain suggested the opening of the lesson; in one answer it was stated that the appropriate period is the first half of the week
- After the phase of discovering

Phases of the lesson - activity:

- Introduction of new subject matter
- Revision
- Problem solving
- Commenting on results of assessment
- When students do not understand or make too many mistakes

Needs:

S: When a student needs help
S: When a group of students does not understand

T: When the teacher needs to get the feedback on students' understanding

Question (iv), Age

In all groups, answers were similar. They are distributed into four domains: form, length, language and frequency of explanations. Respondents' common opinion was that with younger children explanation should be more concrete, shorter but more frequent. It is necessary to use more illustrative forms. It was noted that explanation based on examples from real life situations is more important in case of younger students. Proving was not thought to be acceptable for younger children. The precision and technical level of language was considered more important for older students.

One answer in Group A was: The only difference which depends on students' age is in the level of complexity of the solved problems, there are no differences in any other aspect.

b) ***Student(s) as the transmitter***

In all groups the answers were similar. We present the common ones. Single occurrences in one of Groups A, B or C are mentioned separately.

Question (i), Goals

T: Easy opportunity to discover students' misunderstandings and locate mistakes
T: Possibility to follow students' argumentation in details, getting to understand it
T: Involving students in the work, increasing their motivation
T: Learning students' preferred language
S: Development of students' creativity, argumentation, independence, ability to draw conclusions
S: Forming working skills
S: Development of interaction and communication skills
S: Increasing students' self-confidence

Consolidation and verification of knowledge were explicitly expressed exclusively in Group A. Therefore it seems it is typical for mathematics.

Note: The role in institutionalisation (Brousseau, 1997) of knowledge was not mentioned.

Question (ii), Pitfalls

T: Considerable time requirement, difficult (impossible) lesson planning and management of student work which consequently demands improvisation
T: Necessity to handle individual mistakes and inaccuracies
T: Misunderstanding between students and the teacher concerning explained items
S: Imbalance in students' involvement, passivity of some of them
S: Negative impact of presented mistakes etc.
S: Lesser use of previous experience and existing structures

In Groups A and C, there were two respondents who did not see any pitfall and stressed the positive aspects: "Even an incorrect explanation is an asset.", "Process is equally conductive, or even more, than the result."

Question (iii), Phases of teaching process

The answers were in accordance with the answers from the case a). In Group A, the answer "anytime" was the most frequent (6/17). In Group C, the help of one student to the others was expressed as the phase criterion.

Question (iv), Age

The age differences concerned the level of using visualisation and modelling, the language and the use of concrete examples, all of them simpler in case younger students. The use of the case b) (student(s) as transmitters) was strongly recommended for older students. The independence in selecting the explanation form and the level of abstraction was considerably emphasised in Group A. In one case in group A, hints were proposed as a suitable form of explanation at all ages.

3.1.2. Practical perspective

Both cases a) (10 questions) and b) (7 questions) are analysed separately, only one question concerns both cases. All questions ask for the respondent's own practice.

a) ***Teacher as the transmitter***

The first group of questions deals with phases of lessons, with educational process, and forms used by the respondent and the frequency of using them; these questions are directly linked to theoretical perspective. The second group of questions deals with planning the use of explanation, use of erroneous explanation, demanded precision of the language and consequences of too rare or too frequent use of explanations.

In the answers to the first group of questions, theoretical perspective was stressed mainly by respondents in Groups B and C. This suggests that the answers in the Theoretical part were based on their practical experience. The practical part does not include all the listed forms. The respondents selected visualisation, modelling and illustration, use in other situations, rephrasing and (surprisingly) use of hints.

Answers to the second group of questions brought new information. All teachers but one (Group C) indicated that they choose the place for using an explanation when planning the lesson; their decision is based on their experience. The possibility of increasing the number of explanations according to the situation in the class was explicitly expressed by two teachers in Group A. The need for precision of explanation by the teacher was commonly agreed upon. Only one teacher explicitly mentioned the benefits of decreasing precision to help weaker students.

Considerable was diversity in answers related to deliberate use of explanations containing error. In Group A, 5/17 respondents do not use any form of such an explanation, the others use it for emphasis of more difficult places, to attract students' attention, to break students' acceptance of facts without understanding; two teachers do not use such explanation before "sufficiently long practice" of the subject matter. In Group B explanation containing error is used, while in Group C this question was not answered. The deliberate use of explanation containing error looks to be specific for mathematics (and natural sciences).

As to the frequency of using explanation, all respondents mentioned the danger of students' passivity in the case of its overuse and building incorrect knowledge if explanation is used only very rarely.

b) ***Student(s) as the transmitter***

The questions cover the nature of the demand for explanation (spontaneous or on the teacher's invitation), erroneous explanations (their handling or the possibility of their further use) and the frequency of incorporating explanations in the lessons. Teachers use both spontaneous and invited explanations; all of them praise the advantages of spontaneously proposed explanations. (Typical answer: "Let the person who has something to say speak, all will learn from the mistakes that will occur.") As far as correction of mistakes is concerned, discussion on the mistakes (mediated by the teacher if necessary) is regarded as the appropriate solution. (Typical answer: "We all learn from mistakes, and why it is a mistake.") As far as frequency is concerned, the danger of frequent use of explanation was mentioned in all answers only in such a case when pupils merely listen to this explanation and do not take active part in it. In Groups A and C, also the danger of underdeveloped communication skills and passivity of students was mentioned in case of insufficient use of explanation. The frequency of incorporating explanation turned out to be age dependent. .

4. Concluding remarks

The study confirmed the diversity in approaches to explanation in teaching/learning of mathematics. The sizes of the participating groups of teachers did not allow detection of positive and negative impacts of the use of explanation in teaching/learning processes. There are further questions broadening and deepening our understanding of the position of their use:

The presented results are based on the analysis of teachers' responses to questionnaires. Additional information can be obtained if we combine the questionnaires with observations of real lessons, either directly in the classrooms or watching and discussing video recordings from concrete lessons. These activities offer more information about frequency and location of use of explanation, the language of explanations, differences in the used types of explanations with respect to the level of students' performances, motivational power of explanation in various situations etc.

The use of explanations in teaching should not be restricted to its isolated occurrences. A study on how teachers combine various types of explanations in their lessons and what their relationships to various effects studied in the Theory of Didactical Situations in Mathematics (e.g. Topaze effect, Jourdain effect, use of analogy) (Brousseau, 1997) are offers additional information on explanation in teaching.

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APPENDIX: EXPLANATION - QUESTIONNAIRE

In the questionnaire, two cases are distinguished: a) explanation by the teacher, b) explanation by the student. The label a) or b) in the items relates to them.

<u>"Theoretical" view</u>

1. Specification of the notion of explanation

   What do you regard as *explanation*? (Underline the types that you regard as explanation; you may also add others that you think are missing.)

   Description, reformulation, definition, proof, presenting a counterexample, commentary, illustration, usage in another situation, analogy, decomposition in subcases, model

2. Placing *explanation* in teaching

   What are its aims?
   a)
   b)

   Which obstacles does it bring?
   a)
   b)

   In which phases of teaching do you consider *explanation* most effective?
   a)
   b)

   How does *explanation* differ with respect to the pupils' age?
   a)
   b)

3. Other remarks:

<u>Your own practice</u>

Case a):

In which phases of teaching process do you include *explanation* (with respect to the pupils' age)?

In which phases of <u>mathematics</u> teaching process do you include *explanation* (with respect to the pupils' age)?

Which types of *explanation* do you use?

Do you regard modelling as *explanation*? If so, do you use it? How often? In which situations?

Do you strive for precise language during *explanation*?

Do you plan the places where you will use *explanation* in advance? If yes, on what basis do you plan it?

Do you in advance prepare the form of *explanation* for different cases? If yes, on what basis do you prepare it?

Do you sometimes use *explanation* containing a mistake? If yes, in which cases?

What do you think are the consequences of frequent use of *explanation* by the teacher?

What do you think are the consequences of rare use of *explanation* by the teacher?

Case b):

In your own practice, do you support spontaneous *explanation* by pupils or do you usually solicit pupils' *explanation*? In the latter case, how do you select pupils for *explanation*?

Which *explanation* do you consider more efficient, pupil's spontaneous or solicited *explanation*?

How do you react in case that other pupils do not accept a correct *explanation* by a pupil?

How do you react to an erroneous *explanation* of a pupil? Do you correct the erroneous *explanation* yourself or do you provoke a discussion among pupils?

Do you use a pupil's erroneous *explanation* in the follow-up pedagogical process? If yes, how?

What do you think are the consequences of frequent use of *explanation* by a pupil?

What do you think are the consequences of rare use of *explanation* by a pupil?

Both cases a) and b):

Which *explanation* do you consider more effective – *explanation* by the teacher or by a pupil?

In which situations?

Why?

You can add your experience (positive or negative) with using *explanation* in teaching mathematics:

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FACULTY OF NATURAL SCIENCES
CONSTANTINE THE PHILOSOPHER UNIVERSITY NITRA
ACTA MATHEMATICA 13

REAL-LIFE TASKS – MORE REALITY, PLEASE!

ANDREAS ULOVEC

ABSTRACT. Since many years, connections with reality are asked in teaching mathematics. Many tasks and word problems are however just having a (more or less) realistic cover, but if you look closely you may discover that neither the context nor the data are realistic, let alone interesting for students ("how much metal do you need for the tin can" – tasks). We will present some examples of such pseudo-realistic tasks, as well as some really realistic tasks developed in the EU-project Math2Earth.

1 Why Real-life tasks?

Since decades, real-life contexts are promoted in mathematics education. Although there is some discussion about the 1:1-transferability of school situations into everyday life situations (see e.g. [1], [2]), there is a general agreement that the reality of situations and data used in mathematics teaching helps the motivation of learners and thereby supports the learning process. Hence authors tend to put in lots of word problems into their text books, using a large variety of contexts, most of them with some real-life connection. On a closer look, quite a few of these contexts have two issues: They are not really realistic (either in their context setting and/or their data), and they are not very interesting to students. We will deal with the first issue in this paper; the second issue will be discussed in [3].

2 Real-life tasks in text books – how real is "real"?

We start with giving four examples of word problems from Austrian text books. Let us mention here that this is not done to criticise a specific text book, or to criticise Austrian text books in general. Tasks like these can be found in any arbitrarily chosen text book from any country. This is merely to demonstrate that "looking real" is not the same as "being real", a difference that students can quite often discern, which can make the difference between being interested and being bored. In each example, we will first state the text, and then shortly raise some of the issues that we found.

Example 1: A tinsmith constructs an open can with height 22 cm, length 10 cm, and width 10 cm. What's the amount of sheet metal (in dm^2) needed if the discard is 6 dm^2 20 cm^2.

Issues:
- ☒ Sheet metal is not usually measured in dm^2
- ☒ The discard would be around 40%, a totally unrealistic number for such a construction process

The project Math2Earth has been funded with support from the European Commission, reference number 141876-2008-LLP-AT-COMENIUS-CMP. This publication reflects the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

Example 2: An advertising agency advises a politician that he would need at least 30 one-minute and 20 three-minute TV-spots to win the next elections. Austrian TV sells one-minute broadcasting slots for 1,000 €, and three-minute slots for 2,500 €. How many one-minute and three-minute slots does he have to choose to win the elections with minimum costs?

Issues:
- ☒ The advice is very unusual and does not shade a positive light on either the intelligence of the advisor or the voters
- ☒ Real costs and efficiency of TV spots are extremely dependent on the time of broadcast – and much more expensive than stated here
- ☒ The context is only of marginal interest for students

Example 3: An airplane flies from Frankfurt to Vienna (660 km) and arrives in Vienna 6 minutes earlier if there is a tailwind of 60 km/h (compared with arrival with no tailwind). How fast does the plane go?

Issues:
- ☒ In aviation, usually miles and knots are used as units
- ☒ The real distance between Frankfurt and Vienna is 715 km
- ☒ Usually one knows the speed of the plane and calculates the time saving. The pilot hardly looks at the watch at arrival and asks "ok, 6 minutes early; how fast were we?"

Example 4: The minute hand of a wrist watch is 2.5 cm long. What distance does the tip of the hand travel in one year?

Issues:
- ☒ The watch would be rather large
- ☒ Why would one want to know that? Is there a mileage program for minute hands?

3 Hopefully more reality here

In the EU-funded project "Math2Earth – Bringing Mathematics to Earth", teams from five European countries produced a variety of teaching materials with the intention of showing realistic applications of mathematics in everyday life, the world of work, science etc. We want to present some examples of these materials to give an impression about how real-life tasks can have a realistic context without being too complex or difficult to solve for students in school. The materials can be found in [4], and also online at [5].

Example 1: A plane flies from Vienna to Dubai (distance 2,450 miles = 4,537 km) with a Boeing 737-800. The average air speed is 400 knots (miles per hour). How much fuel is required, if the minimum fuel is calculated as follows?

1. Fuel for taxiing from the parking position to the runway (200 kg)
2. Fuel for the flight from Vienna to Dubai (2,400 kg/h)
3. 5% of 2. as spare (e.g. to compensate for wind etc.)

4. Fuel for the flight from the destination airport to the alternative airport (distance in this case: 160 miles)
5. Minimum remaining fuel (after the landing there has to be fuel left for 30 minutes of flying)

Figure 1: Plane shortly after departure

Example 2: A study says that about one third of the Austrian energy consumption (58,884 GWh) could be covered by using solar cells. A typical solar cell module is rectangular, 160 cm x 90 cm, and has an average power output of 200 W. The annual average sunshine in Austria is 4.5 hours per day. How many solar cells would be needed if the goal of the study should be achieved? How many km^2 and what percentage of Austria would be covered with solar cells? Is it realistic to do this?

Figure 2: A row of solar cells

Example 3: In the gardens of Schönbrunn castle in Vienna (see map below) the lawn has to be replanted regularly. How much seed is needed, if the producer recommends 1 kg of seed per 15 m^2?

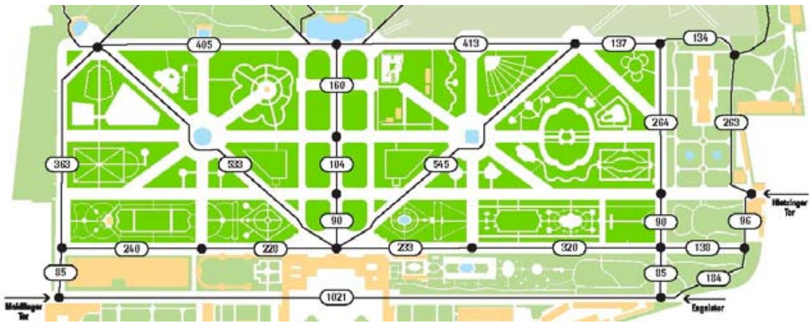

Figure 3: Map of the "large gallery" of the gardens at Schönbrunn castle

4 Summary

Realistic context is a mean of achieving motivation in students. It is of course by far not the only mean, and we do not argue for each and every task, calculation etc. in mathematics teaching having to have realistic (or any) context. But showing students that mathematics *does* have real-life applications, and that it *is* useful for their later lives, can greatly enhance motivation, and thereby enhance learning. And not to forget that doing calculations that have real meaning is just more fun!

References

[1] Jo Boaler: *The Role of Contexts in the Mathematics Classroom: Do They Make Mathematics More "Real"?*, For The Learning of Mathematics 13(2), 12-17, 1993

[2] Jean Lave: *Cognition in practice: Mind, Mathematics and Culture in Everyday Life*, New York, Cambridge University Press, 1988, ISBN 0-521-35015-8

[3] Andreas Ulovec: *MEETING in Mathematics & Math2Earth: Two international co-operations on creativity, motivation, and giftedness*, Proceedings of the Sixth International Conference on Creativity in Mathematics Education and the Education of Gifted Students, in press

[4] John Andersen et al.: *Math2Earth – Bringing Mathematics to Earth*, Prague, PRVOKRUH, 2010, ISBN 978-80-901470-2-7

[5] http://www.math2earth.eu (Accessed at August 11, 2010)

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VIII. Nitrianska matematická konferencia

13. – 17. september 2010

Nitra

PROGRAM KONFERENCIE

<u>13. septembra 2010 – pondelok</u>

Oficiálne otvorenie Letnej školy doktorandov

<u>14. septembra 2010 – utorok</u>

Plenárne prednášky

miestnosť THC 212

8^{30} hod. – 9^{15} hod.
Jarmila Novotna
EXPLANATION IN TEACHING MATHEMATICS. TEACHERS' VIEWS

9^{30} hod. – 10^{15} hod.
Nicolas Mousolides
THE LEARNING OF MATHEMATICS AND MATHEMATICAL MODELLING USING CONCEPTUAL TECHNOLOGICAL TOOLS

10^{15} hod. – 10^{30} hod. Coffee break - miestnosť M – 6

10^{30} hod. – 11^{15} hod.
Freya Hreinsdóttir
ICELANDIC GEOGEBRA INSTITUTE – ON OPEN SOURCE SOFTWARE AND OPEN SOURCE TEACHING MATERIAL

11^{30} hod. – 12^{15} hod.
Maria Luiza Cestari
INTERPLAY BETWEEN HOME-SHOOL FOR NORWEGIAN PUPILS IN THE LEARNING OF MATHEMATICS: A CASE STUDY

10^{15} hod. – 10^{30} hod. Lunch break

13^{30} hod. – 14^{15} hod.
Barbro Grevholm
VARYING VIEWS OF MATHEMATICS AND IMAGES OF MATHEMATICIANS AMONG NORWEGIAN UPPER SECONDARY SCHOOL STUDENTS

14^{30} hod. – 15^{15} hod.

Andreas Ulovec

REAL-LIFE TASK - MORE REALITY, PLEASE!

14^{30} hod. – 15^{15} hod.

Vladimir Georgiev

MATH TO EARTH: EXPERIENCE OF MATH LABS AND ACTIVITIES IN THE YEAR OF GALILEI IN PISA

10^{15} hod. – 10^{30} hod. Coffee break - miestnosť C 204

16^{30} hod. – 17^{15} hod.

John Andersen

MATHEMATICAL OPPORTUNITIES – MATHEMATICAL AWARENESS: A QUESTION OF FOCUS

17^{30} hod. – 18^{15} hod.

Jevgenia Sendova

THE MATHEMATICS IN THE BEAUTIFUL AND THE BEAUTIFUL IN MATHEMATICS

16. septembra 2010 – štvrtok

Plenárne prednášky

miestnosť P 2

9^{00} hod. – 9^{45} hod.

OLEG MUSKAROV, JEVGENIJA SENDOVA, NELI DIMITROVA

ENHANCING THE RESEARCH POTENTIAL OF MATHEMATICALLY GIFTED HIGH-SCHOOL STUDENTS

10^{00} hod. – 10^{45} hod.

KRISTIN BJARNADÓTTIR

HISTORY OF MATHS EDUCATION IN ICELAND

11^{15} hod. – 12^{00} hod.

BOHUMIL NOVÁK

NON-TRADITIONAL MATHEMATICS TASKS AND ACTIVITIES IN ELEMENTARY SCHOOL EDUCATION

12^{15} hod. – 13^{00} hod.

HENK VAN DER KOOIJ

MATHEMATICS AT WORK

Rokovania v sekciách

Sekcia 1, miestnosť C 212

Rokovanie vedie : PaedDr. Gabriela Pavlovičová, PhD.

14^{00} hod. – 14^{15} hod.

JOANNA MAJOR, ZBIGNIEW POWĄZKA

FROM RESEARCHES UPON SOLVING STEREOMETRIC TASKS BY STUDENTS

14^{15} hod. – 14^{30} hod.

ADAM CZAPLIŃSKI, MACIEJ MAJOR

EXCEL SPREADSHEET AS A SUPPORTING TOOL FOR CONDUCTING MATHEMATICAL REASONING

14^{30} hod. – 14^{45} hod.

MAREK POMP, ZUZANA VÁCLAVÍKOVÁ

KONCEPCE TVORBY INTERAKTIVNÍCH ÚLOH A TESTŮ V MATEMATICE

14^{45} hod. – 15^{00} hod.

MAREK POMP, ZUZANA VÁCLAVÍKOVÁ

FAKTORY OVLIVŇUJÍCÍ GEOMETRICKOU PŘEDSTAVIVOST ŽÁKŮ ZŠ

15^{00} hod. – 15^{15} hod.

DARINA STACHOVÁ

GEOMETRICKÁ GRAMOTNOSŤ ŠTUDENTOV VŠ TECHNICKÉHO ZAMERANIA

15^{15} hod. – 15^{30} hod.

IVETA SCHOLTZOVÁ

PRVÉ KROKY K MATEMATICKEJ GRAMOTNOSTI

15^{30}hod. – 15^{45} hod.

VIERA UHERČÍKOVÁ, PETER VANKÚŠ

E-LEARNINGOVÝ KURZ: NETRADIČNÉ METÓDY VO VYUČOVANÍ MATEMATIKY

15^{45}hod. – 16^{00} hod.

VERONIKA ZEĽOVÁ

ŽIVOT ŽIAKA PRIMÁRNEJ ŠKOLY AKO VÝCHODISKO PRE TVORBU ÚLOH ROZVÍJAJÚCICH MATEMATICKÚ GRAMOTNOSŤ

16^{00}hod. – 16^{15} hod.

ALENA PRÍDAVKOVÁ

PRÍSTUPY K RIEŠENIU JEDNEJ MATEMATICKEJ ÚLOHY V PRÍPRAVE UČITEĽOV ELEMENTARISTOV

16^{15}hod. – 16^{30} hod.

JANA PŘÍHONSKÁ

ROZVOJ LOGICKÉHO MYŠLENÍ ŽÁKŮ

Sekcia 2, miestnosť C 217

Rokovanie vedie : PaedDr. Marek Varga, PhD.

14^{00} hod. – 14^{15} hod.

RŮŽENA BLAŽKOVÁ

ŽÁCI SE SPECIFICKÝMI VZDĚLÁVACÍMI POTŘEBAMI

14^{15} hod. – 14^{30} hod.

IRENA BUDÍNOVÁ

VÝVOJ POJMU FUNKCE OD ARCHIMEDA PO NEWTONA

14^{30} hod. – 14^{45} hod.

IVETA KOHANOVÁ

METÓDA PROBLEM SOLVING V PRÍPRAVE BUDÚCICH UČITEĽOV MATEMATIKY

14^{45} hod. – 15^{00} hod.

HYČKOVÁ SILVIA – KONTROVÁ LÝDIA

FUNKCIA VIZUALIZÁCIE V MATEMATICKOM VZDELÁVANÍ

15^{00} hod. – 15^{15} hod.

MAREK MOKRIŠ

C.a.R A METÓDA GENEROVANÝCH PROBLÉMOV

15^{15} hod. – 15^{30} hod.

RADEK KRPEC

PROČ JE DŮLEŽITÉ UVĚDOMOVAT SI VZTAH MEZI PRAVDĚPODOBNOSTMI OPAČNÝCH JEVŮ

15^{30}hod. – 15^{45} hod.

PETER VANKÚŠ, EMÍLIA KUBICOVÁ

POSTOJE ŽIAKOV 5. A 9. ROČNÍKA ZŠ K MATEMATIKE

15^{45}hod. – 16^{00} hod.

MILAN STACHO, MÁRIA BRANICKÁ

VYUČOVANIE MATEMATIKY V DRUHOM STUPNI ŠTÚDIA NA FAKULTE PEDAS ŽILINSKEJ UNIVERZITY V ŽILINE

16^{00}hod. – 16^{15} hod.

ĽUBOMÍR CZANNER – JÁN ČIŽMÁR

VÝSTAVBA ZÁKLADNÝCH ARITMETICKÝCH ŠTRUKTÚR V PRIMÁRNEJ ŠKOLE NA SLOVENSKU, V NEMECKU A VO VEĽKEJ BRITÁNII

16^{15}hod. – 16^{30} hod.

JAROSLAV PERNÝ - JANA HANKOVÁ - TEREZA NOVÁKOVÁ - TEREZA VOTRUBCOVÁ

BUDOUCÍ UČITELÉ A ROZVOJ GEOMETRICKÉ PŘEDSTAVIVOSTI ŽÁKŮ

16^{45} hod. miestnosť C 117 a C 118

Krátke spoločenské stretnutie účastníkov konferencie

<u>17. septembra 2010 - piatok</u>

PLENÁRNE PREDNÁŠKY

miestnosť C 212

9^{00} hod. - 9^{45}

Beáta Stehlíková

PRIESTOROVÁ ŠTATISTIKA

Rokovania v sekciách

Sekcia 1 Miestnosť THC 212

Rokovanie vedie : PaedDr. Gabriela Pavlovičová, PhD.

9^{45} hod. – 10^{00} hod.

JAROSLAV BERÁNEK, JAN CHVALINA

INVARIANTNÍ PODGRUPY GRUP OBYČEJNÝCH LINEÁRNÍCH DIFERENCIÁLNÍCH OPERÁTORŮ DRUHÉHO ŘÁDU

10^{00} hod. – 10^{15} hod.

OLEG PALUMBÍNY - RÓBERT VRÁBEĽ

ON EXISTENCE OF OSCILLATORY SOLUTIONS OF BINOMIAL FOURTH-ORDER LDEs

10^{15} hod. – 10^{30} hod.

OLEG PALUMBÍNY

ALTERNATÍVNE ZAVEDENIE OBORU VŠETKÝCH NEZÁPORNÝCH REÁLNYCH ČÍSEL

10^{30} hod. – 10^{45} hod.

ANTONIO BOCCUTO - XENOFON DIMITRIOU - NIKOLAS PAPANASTASSIOU

CONVERGENCE THEOREMS FOR THE OPTIMAL INTEGRAL IN RIESZ SPACES

10^{45} hod. – 11^{00} hod.

PETER VRÁBEL

SÚVIS RIEMANNOVSKEJ INTEGROVATEĽNOSTI FUNKCIE S EXISTENCIOU LIMÍT V BODOCH JEJ DEFINIČNÉHO OBORU

11^{00} hod. – 11^{15} hod. Coffee break - miestnosť M – 6

11^{15} hod. – 11^{30} hod.

JOZEF FULIER

NIEKOĽKO POZNÁMOK O SUMÁCII ČÍSELNÝCH RADOV

11^{30} hod. – 11^{45} hod.

ĽUBOMÍR RYBANSKÝ, MARTA VRÁBELOVÁ

ŠTATISTICKÉ SPRACOVANIE VÝSLEDKOV VÝSTUPNÉHO TESTU PRE 5. ROČNÍK PROJEKTU KEGA 3/7001/09

11^{45} hod. – 12^{00} hod.

JÚLIUS JENIS, MARTA VRÁBELOVÁ

STOCHASTIKA V ŠTÚDIU UČITEĽSTVA PRE PRIMÁRNE VZDELÁVANIE

12^{00} hod. – 12^{15} hod.

GABRIELA PAVLOVIČOVÁ, JÚLIA ZÁHORSKÁ, ĽUBOMÍR RYBANSKÝ

ANALÝZA RIEŠENIA VYBRANÝCH ÚLOH V TESTOVANÍ ŽIAKOV 5. ROČNÍKA ZŠ

12^{15} hod. – 12^{30} hod.

SOŇA FÁNDLYOVÁ – MIROSLAVA SOVIČOVÁ

PRINÍPY TVORBY ÚLOH PROJEKTU LLP COMENIUS COMPASS (UKÁŽKY ÚLOH VYTVORENÝCH V HOLANDSKU)

12^{30} hod. – 12^{45} hod.

KITTI VIDERMANOVÁ – JANKA MELUŠOVÁ

VYUŽITIE GEOGEBRY PRI RIEŠENÍ OPAČNÝCH KONŠTRUKČNÝCH ÚLOH

Sekcia 2, miestnosť C 217
Rokovanie vedie : RNDr. Dušan Vallo, PhD.

9^{45} hod. – 10^{00} hod.
ONDREJ ŠEDIVÝ – DUŠAN VALLO
POUŽITIE POMERU PRI RIEŠENÍ GEOMETRICKÝCH ÚLOH (METODICKÝ PRÍSPEVOK)

10^{00} hod. – 10^{15} hod.
ADAM PŁOCKI
OVĚŘOVÁNÍ HYPOTÉZ A MATEMATIZACE VE STOCHASTICE PRO UČITELE

10^{15} hod. – 10^{30} hod.
SOŇA ČERETKOVÁ - ĽUBICA KORENEKOVÁ - JANKA MELUŠOVÁ
ÚLOHY (NIELEN) O ŽIAROVKÁCH A ÚSPORE ELEKTRICKEJ ENERGIE

10^{30} hod. – 10^{45} hod.
ZOLTÁN FEHÉR
PRAVDEPODOBNOSŤ ALEBO ŠANCA?

10^{45} hod. – 11^{00} hod.
JÁN ŠUNDERLÍK
POSTOJE V PRÍPRAVE ŠTUDENTOV UČITEĽSTVA MATEMATIKY - PRÍPADOVÁ ŠTÚDIA

11^{00} hod. – 11^{15} hod. Coffee break - miestnosť C 204

11^{15} hod. – 11^{30} hod.
MAREK VARGA, MICHAELA KLEPANCOVÁ
LIMITA POSTUPNOSTI A JEJ MOTIVÁCIA

11^{30} hod. – 11^{45} hod.
ALENA VIZIOVÁ
VYUŽITIE PROGRAMU WINPLOT VO VYUČOVANÍ FUNKCIÍ A ICH VLASTNOSTI (VÝSLEDKY EXPERIMENTU)

11^{45} hod. – 12^{00} hod.
FILIP HALAMA – PATRÍCIA BENICKÁ
JEDEN Z PRÍSTUPOV ROZVÍJANIA MATEMATICKÉHO MYSLENIA V OBLASTI KOMBINATORIKY

12^{00} hod. – 12^{15} hod.
DUŠAN VALLO – ONDREJ ŠEDIVÝ – VILIAM ĎURIŠ
FLEXIBILNÉ MNOHOSTENY VO VÝUČBE STEREOMETRIE

12^{15} hod. – 12^{30} hod.

DANA MITRENGOVÁ, TOMÁŠ LENGYELFALUSY

DÔKAZY VZŤAHOV PRE SÚČTY HODNÔT FUNKCIÍ ARCUS

12^{30} hod. – 12^{45} hod.

EVA BARCÍKOVÁ

MOTIVÁCIA POMOCOU RIEŠENIA KONŠTRUKČNÝCH GEOMETRICKÝCH ÚLOH S VYUŽITÍM IKT

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OBSAH

Názov:	ACTA MATHEMATICA 13, zväzok 2
Vydavateľ:	Fakulta prírodných vied UKF v Nitre
Zostavovatelia:	prof. RNDr. Ondrej Šedivý, CSc. doc. PaedDr. Soňa Čeretková, PhD. PaedDr. Janka Melušová, PhD. RNDr. Dušan Vallo, PhD. RNDr. Kitti Vidermanová, PhD.
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