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# ANALYSIS OF THE WINNING STRATEGY OF THE GAME ENADES AS A TASK FOR PUPILS

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**ABSTRACT.** *In this paper we will deal with the mathematical didactical game Enades. Analysis of its winning strategy is task that can be solved with student during mathematics education at secondary schools and also at the universities. In the paper we will introduce the game, analysis of the strategy of two versions of this game and also our experience from using this activity in the education in the preparation of future mathematics teachers.*

**KEY WORDS:** *mathematics education, didactical game, winning strategy*

**ABSTRAKT.** *V článku sa zaoberáme matematickou didaktickou hrou Enády. Analýza výhernej stratégie tejto hry je úloha, ktorá môže byť riešená v rámci matematického vzdelávania na druhom stupni základnej školy, na strednej škole ale aj na vysokej škole. V článku predstavíme hru, analýzu výherných stratégií dvoch verzií hry a tiež naše skúsenosti z používania tejto aktivity v rámci prípravy budúcich učiteľov matematiky.*

**KĽÚČOVÉ SLOVÁ***: matematické vzdelávanie, didaktická hra, výherná stratégia*

**CLASSIFICATION:** *D40, A20, E30*

## Introduction

The game is activity that is attractive for a majority of pupils and also a lot of adults. Attractiveness and motivational impetus of the game are used also in the mathematics education (Burjan and Burjanová, 1991; Kohanová, 2010; Pavlovičová and Švecová, 2009; Pavlovičová et all, 2012; Slavíčková, 2008; Vallo, Záhorská and Ďuriš, 2011; Vallo and Šedivý, 2012; Vidermanová and Uhrinová, 2011). The game used by this way we call didactical game or mathematical didactical game.

In our paper we will introduce a game from our book *Didaktické hry v matematike* (Didactical games in mathematics; Vankúš, 2012). This book is dealing with the problematic of mathematical didactical games. It is accessible for free at the address http://www.comae.sk/didaktickehry.pdf, therefore is available for the teachers community. The important part of the book is collection of 30 didactical games designed for various thematic areas of the secondary school mathematics education.

The game *Enades* from this collection is the content of our paper. We will describe brief analysis of its winning strategy as a task for pupils. We will also speak about our experience of using this activity in the preparation of future mathematics teachers.

This paper is meant to be motivation for readers to use analysis of winning strategy of didactical games as activity in mathematics education.

## Mathematical didactical game Enades

In this chapter we will present two versions of the game Enades. They are different just in the rules so all other things in the game description are the same for both versions. First we will give some basic information about the didactical game Enades applicable for secondary school mathematics:

Thematic area: This game is suitable for the thematic area of Powers and Roots.

Educational targets: To practice determining the powers of some numbers by heart. To get feedback on pupils' knowledge from the subject matter. The game develops pupils' combinatorial and strategic thinking.

Game environment: Pupils and the teacher: Pairs of pupils at desks play. The teacher has an organizational and controlling role.

Material environment: Sheet of paper for each pair.

Game duration: 5 -10 min

Benefits of the game: Active work of the class, inner motivation of pupils through competitiveness. The pupils' mutual cross-checking eases the load on the teacher.

Game rules: First we will present the version of the game Enades as published in book (Vankúš, 2006, p. 45). Then we will present the version from the book (Vankúš, 2012, p. 119).

**Enades version 1 (Vankúš, 2006, p. 45)**

Game procedure: The players deduct any powers of 2, 3 or 5 with an exponent of a non-negative integer from an initial number $n$ (e.g. $n = 100$). Both players take turns and write down the status of the game on the sheet of paper. The player who gets 0 in his/her turn wins. In the following game, the pupils will change the order in which they started the previous game. Players play more games. The lowest number is 2 to make sure that each player has started the same number of games. Pupils will put down the mutual score and submit the record to the teacher. Both the winner and the loser will get a certain number of points for the activity for each game (e.g. 3 points for the winner, 1 point for the loser). A sample game – see figure 1.

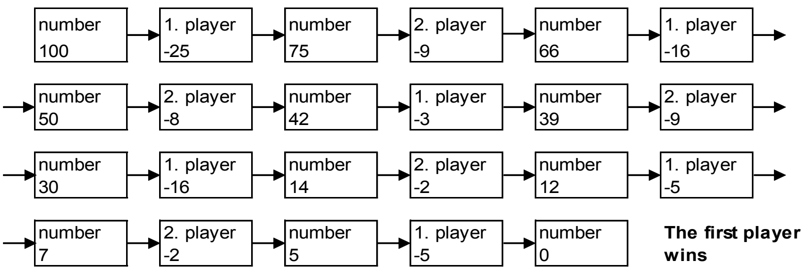


Figure 1: A sample Enades game

**Enades version 2 (Vankúš, 2012, p. 115)**

Game procedure: In the second version of the game the players deduct any powers of 2, 3 or 5 with an exponent of a positive integer from an initial number $n$ (e.g. $n = 100$). Both players take turns and write down the status of the game on the sheet of paper. The player who gets 0 or 1 in his/her turn wins. All other things are the same as in version 1.

So the differences between the versions are that in version 1 we use non-negative integers as powers, in version 2 we use positive integers. So in version 1 we can also deduct number 1, which we get when we power any number by 0. Winning condition in version 1 is to get 0; in version 2 winning numbers are 0 and 1.

## Analysis of the game winning strategy

Before the analysis of the game we let pupils play few games to better understand game mechanics. Just after they played some games we can start with the analysis. Analysis of the game winning strategy differs for version 1 and version 2 of the game, so we will briefly describe them separately.

### Winning strategy Enades version 1

Pupils will find out, that some positions in this game are winning and some are losing after some thinking during the playing of the game. Because the winning final position is 0, any number from which we can get to 0 is winning. So, all the non-negative integer powers of 2, 3 and 5 are winning positions. So they are numbers: 1, 2, 3, 4, 5, 8, 9, 16, 25, 32, 64 and 81.

We can see that all numbers from 1 to 5 are winning positions. When the player is on the number 6, any possible move he/she makes will put the next player on one of these winning positions. So the number 6 is losing position. Any number from which we can get to the number 6 by deducting non-negative integer power of 2, 3 and 5 is again winning position. Let us depict the table with the new winning positions added – see figure 2.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | |
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | |
| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | |
| 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

Figure 2: An analysis of winning strategy Enades v. 1

As we can see new losing position is 12. Pupils will soon find out that the next losing position will be 18, 24, and so on. This fact is easy to notice, just looking on some numbers those we can use in the game: 1, 2, 3, 4, 5, 8 and 9. We can see that just 6 and 7 are the numbers from 1 to 10 those are not powers we can deduct in the game. But we can go from 7 to 6 by subtracting 1. So just every number divisible by 6 can be losing position in the game Enades version 1. Any other power we can use in the game is not divisible by 6, so all number divisible by 6 will be the losing positions. So the winning strategy for the first player in the game will be to start e.g. with subtracting 4 to get to 96. That is losing position. Now when second players moves we again do the move so he/she will be on the nearest losing position – number divisible by 6. By this way the first player will win.

### Winning strategy Enades version 2

The process of analysis of the winning strategy is the same as for version 1. We will try to find the winning and losing positions. In this version of the game we cannot subtract 1. Initial winning positions are 0 and 1. Any number from those we can get to 0 or 1 by subtracting available powers will be winning position. So they are numbers: 1–6, 8–10, 16, 17, 25–28, 32, 33, 64, 65, 81 and 82.

We can see that in this version the first losing position is number 7. The new winning positions we get by adding all available powers to number 7. Then we get the next losing position, number 13. And so on we can apply this procedure to get all the losing positions in the game. The final table with all the losing positions is in the figure 3.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | |
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | |
| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | |
| 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |

Figure 3: All losing positions, Enades v. 2

In this case the losing positions are not to be found as easy as in version 1. It is because more difficult mechanics of this version of the game. The winning strategy in this version is for the second player, because the initial number 100 is the losing position. After the first move the second player subtracts number so that he/she gets the losing position number. Then he/she after every move of the first player just subtract some of available powers so that he/she gets losing position number. So the first player is after each round in the losing position and he/she loses the game.

## Experience from analysing the games with the future mathematics teachers

We have tried to make this activity with the future mathematics teachers those were in the bachelor degree of their study. They had to analyse nine mathematical didactical games: 3D Noughts and Crosses, Bard, Dim, Enades, Snake, Powers, Letter 'L' Travelling, Number of Divisors and Equations (Vankúš, 2012). They had to find the answers to these questions:

1. Has in the game advantage the first player or the second player? Why?
2. Is there a strategy to make a draw for any of players? What is it?
3. Is there a winning strategy for any players? What is it?

They got also additional instructions: The answers on these questions explain and illustrate on concrete examples. If it is difficult to find the answers, try to take smaller game board or fewer objects in the game. Write the answers to the questions for your board dimension or your fewer objects and state also that dimension or number of objects.

The activity was team work; there were 2 teams of 5 members. They had 4 hours for the activity. We will now speak about their result more in detail.

The first group for the question 1 stated concrete scores of two players. So they started their argumentation with examples from games. Further they used in argumentation for this question more mathematically based facts as for game 3D Noughts and Crosses: "Nor the first or second player has the advantage. There are $4^3$ possibilities so the chance to win is equal." Or for the game Bard the first team wrote: "The advantage from the first move is compensated by the fact that first player needs to make the numbers divisible by 3 so he has just one third of numbers available. So there is no advantage for the first neither for the second player. What matter is the numbers they use at the beginning and how they use them." The most exact answer had the first group for the question 1 for the game Equations: "The advantage is given by the number of equations. For the odd number the second player wins, for the even number wins the first player."

The second group used for the question 1 also some observation-based arguments. For the game 3D Noughts and Crosses they wrote: "It comes from our observation that the first player has the advantage. When he/she plays optimal he/she wins." For the game Powers they used incomplete analysis: "The second player has the advantage. We tried almost all possible first moves looking if the second player could win and he can win in all of them."

The question 2 about the draw strategy was reduced by students from both teams just to the question if the draw can be in the games or not. They all gave correct answers to this modified question except for the second group for the game 3D Noughts and Crosses, where they answered that there is no possible draw but the correct answer is that draw is possible in this game. The argumentation for this question was usually simple, they just answered yes or no or used logical argument as the first group for the game Powers: "The draw is not possible. Just one player can come to the negative value as the first."

The question 3 about the winning strategy was the most difficult. The groups wrote some strategies those were just playing tips e.g. for the game 3D Noughts and Crosses the second team stated: "The strategy is to start moves in the middle, then we have more possibilities to win and the second player cannot defend all of them." or for the game Snake the first group wrote: "The strategy is to take as much space as possible and to make for the opponent impossible to place his/her snake in the game board." But some answers were very nice and mathematics based. The second group for the game Bard said: "The first player puts 3 to the middle and then he/she just controls the divisibility. The game we can analyze more easy when we use instead of original number 1,…,9 just their remnants after division by 3, so we use just number 0, 1 and 2, each three times." At the end of the activity the first groups did good analysis of the tactics of the games 3D Noughts and Crosses, Bard, Dim and Number of Divisors and found exact strategy of the game Equations. The second group did good analysis of the tactics of the games 3D Noughts and Crosses, Bard and found exact strategy for the games Dim and Enades. After the activity we discussed the answers with the students and we spoke about their good or false results.

The activity was very successful; both teams managed to find some nice answers with good argumentation. They liked the activity and found it good to get know selected mathematical didactical games that can be useful for their teaching practice.

## Conclusion

In our paper we dealt with mathematical didactical game Enades. We discussed two versions of this game and found the winning strategy for both. Then we wrote about the analysis of the games strategy as the activity for future mathematics students. We described our experience from this activity and their results and feedback.

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