FACULTY OF NATURAL SCIENCES
CONSTANTINE THE PHILOSOPHER UNIVERSITY NITRA
ACTA MATHEMATICA 16

I – CONVERGENCE AND I - CONTINUITY OF THE FUZZY NUMBER-VALUED FUNCTIONS

I – KONVERGENCIA A I – SPOJITOSŤ FUNKCIÍ S HODNOTAMI VO FUZZY ČÍSLACH

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ABSTRACT. *In this paper we study a convergence and continuity of the fuzzy number-valued functions with respect to an ideal. We prove some basic properties this convergence and continuity.*

KEY WORDS: *fuzzy number, I-convergence, I-continuity*

ABSTRAKT. *Tento článok pojednáva o konvergencii a spojitosti funkcií s hodnotami vo fuzzy číslach vzhľadom na nejaký ideál. V článku sú dokázané základné vlastnosti takejto konvergencie a spojitosti.*

KĽÚČOVÉ SLOVÁ*: fuzzy číslo, I-konvergencia, I-spojitosť*

CLASSIFICATION: *26A03 26A15, 54A20, 03E72, I19*

Introduction

I-convergence and *I*-continuity of the real-valued functions was defined by T. Šalát, P. Kostyrko and W. Wilczyński in [2]. It is possible find further results and some generalizations of this problem in papers [1], [3], [5]. We generalize *I*-convergence and *I*-continuity for the fuzzy number-valued functions. The fuzzy set-valued mappings are studied in various settings in the last few years. For example the integrals of fuzzy set-valued mappings have applications in mathematical economics and optimal control theory.

I-convergence of fuzzy numbers

In this section we deal with the structure of fuzzy numbers and its *I*-convergence.

Definition 1. The fuzzy number is any function $u : R \to [0,1]$, where *R* is the set of real numbers, satisfying the following conditions:

(1) there exists $x_0 \in R$ such that $u(x_0) = 1$,

(2) the $\alpha -$cut set $(u)^\alpha = \{x \in R; u(x) \geq \alpha\}$ is convex for every $\alpha \in (0,1]$,

(3) u is upper semi-continuous, i.e. any α - cut $(u)^\alpha$ is a closed subset of R,

(4) the support $\overline{\{x \in R; u(x) > 0\}}$ of the function u is a compact set.

The set of fuzzy numbers we denote *E*. The set of real numbers can be embedded into *E*; the real number *z* is identified with the fuzzy number $\bar{z} = \chi_{\{z\}}$, i.e. with the function

$$\chi_{\{z\}}(x) = \begin{cases} 1, & x = z \\ 0, & x \neq z. \end{cases}$$

For the proof of the following lemma see [4].

Lemma 2. If $u \in E$ then

(a) $(u)^\alpha$ is a closed interval for every $\alpha \in (0,1]$,

(b) $(u)^{\alpha_2} \subset (u)^{\alpha_1}$ whenever $0 \le \alpha_1 \le \alpha_2 \le 1$,

(c) if $\alpha_n \uparrow \alpha$, then $\bigcap_{n=1}^{\infty} (u)^{\alpha_n} = (u)^{\alpha}$.

Conversely, if system of intervals $\{(M)^\alpha ; \alpha \in [0,1]\}$ fulfills (a) – (c), then there exists a unique $u \in E$ such that $(u)^\alpha = (M)^\alpha$ for every $\alpha \in (0,1]$.

The sum of fuzzy numbers *u*, *v* is a fuzzy number *z* such that

$$z = u + v \iff (z)^\alpha = (u)^\alpha + (v)^\alpha$$

for every $\alpha \in (0,1]$, where the sum of intervals $[a,\ b] + [c,\ d] = [a + c,\ b + d]$.

The partial ordering on the set *E* is defined in the following way

$$u \le v \iff (u^\alpha) \le (v)^\alpha$$

for every $\alpha \in (0,1]$, where $[a,b] \le [c,d] \iff (a \le c \wedge b \le d)$.

The Hausdorff distance *d* of closed possibly degenerate intervals is defined by equation

$$d([a,b],[c,d]) = \max\{|c-a|,|d-b|\}.$$

We can define the metric $D : E \times E \to [0,\infty)$,

$$D(u,v) = \sup\{d((u)^\alpha,(v)^\alpha); \alpha \in (0,1]\}.$$

Then (E,D) is a complete metric space. The following properties of the metric *D* can be found in [6]:

(i) $D(u+w, v+w) = D(u,v)$ for all $u,v,w \in E$,

(ii) $D(u+v, \mathrm{w}+z) \le D(u,w) + D(v,\mathrm{z})$ for all $u,v,w,z \in E$,

(iii) $D(u+v,\bar{0}) \le D(u,\bar{0}) + D(v,\bar{0})$ for all $u,v \in E$,

(iv) $D(u+v,\mathrm{w}) \le D(u,w) + D(v,\bar{0})$ for all $u,v,w \in E$.

The product of fuzzy numbers *u*, *v* is a fuzzy number *z* such that

$$z = u \cdot v \Leftrightarrow (z)^\alpha = (u)^\alpha \cdot (v)^\alpha \text{ for every } \alpha \in (0,1],$$

where the product of intervals $[a,b] \cdot [c,d] = \{xy; x \in [a,b] \wedge y \in [c,d]\}$. The absolute value of a fuzzy number u is a fuzzy number $|u|$ such that $(|u|)^\alpha = |(u)^\alpha|$ for every $\alpha \in (0,1]$, where $|[a,b]| = \{|x|; x \in [a,b]\}$.

For any intervals $[a,b],[c,d],[e,f]$

$$d([a,b]\cdot[e,f],[c,d]\cdot[e,f]) \le \max\{|e|,|f|\}. d([a,b]\cdot[c,d]).$$

Let $u, v, z \in E$ and let $|z| \leq K$, $K \in R^+$. Then for every $\alpha \in (0,1]$ $(|z|)^\alpha \subseteq [0, K]$ and

$$d((uz)^\alpha, (vz)^\alpha) \leq d((Ku)^\alpha, (Kv)^\alpha) = Kd((u)^\alpha, v^\alpha),$$

consequently $D(uz, vz) \leq KD(u, v)$.

Lemma 3. If a sequence $\{u_n\}_{n=1}^\infty$ of fuzzy numbers converges in metric space (E, D) to $u \in E$, then there is $K \in R$ such that $u_n \leq K$ for every $n \in N$.

Proof. There is $m \in N$ such that $D(u_n, u) < 1$ for any $\in N, m < n$. Denote

$$(u_n)^\alpha = \left[a_n^{(\alpha)}, b_n^{(\alpha)}\right], u^{(\alpha)} = \left[a^{(\alpha)}, b^{(\alpha)}\right],\ n \in N.$$

For any $\alpha \in (0,1]$ we have

$$\left|b_n^{(\alpha)} - b^{(\alpha)}\right| \leq d((u_n)^\alpha, (u)^\alpha) \leq D(u_n, u) < 1, m < n.$$

Put

$$L = \sup\left\{\max\left\{b_1^{(\alpha)}, b_2^{(\alpha)}, \cdots, b_m^{(\alpha)}\right\};\ \alpha \in (0,1]\right\}, M = \sup\left\{b^{(\alpha)} + 1; \alpha \in (0,1]\right\}.$$

The number K, $K = \max\{L, M\}$, is searched number.

I-convergence of a sequence of fuzzy numbers is defined by notion admissible ideal I of sets of natural numbers.

Set I, $I \subseteq P(N)$, where $P(N)$ is system of all subsets of N, are called ideal, if satisfies conditions:

1^0 for any $A \in I, B \in I$ also $A \cup B \in I$;

2^0 if $A \in I$ and $B \subseteq A$, then $B \in I$;

3^0 every finite subset of N belongs to I;

4^0 $N \notin I$.

Let N_p denote the set of all even natural numbers. Let $I_1 = \{A \in P(N); A \text{ is finite}\}$, $I_2 = I_1 \cup \left\{A \in P(N);\ \exists B \in P\left(N_p\right) \exists C \in I_1\ A = B \cup C \text{ and } B \text{ is infinite}\right\}$. Then I_1, I_2 are examples of admissible ideals.

It is suitable to use also the notion of the filter which is determined by an ideal I, i. e. $\mathcal{F}(I) = \{X \in P(N);\ \exists\ A \in I\ X = N - A\}$.

Next assertions about an ideal I are evident:

5^0 if $A \in I$, then $N - A \notin I$,

6^0 if $A \in I$, then $N - A$ is infinite.

Definition 4. A sequence $\{u_n\}_{n=1}^\infty$ of fuzzy numbers is said to converge to u with respect to the ideal I (we write $I - \lim_{n\to\infty} u_n = u$) if

$$A(\varepsilon) = \{n \in N; D(u_n, u) \geq \varepsilon\} \in I$$

for each $\varepsilon > 0$.

Any sequence $\{u_n\}_{n=1}^\infty$ of fuzzy numbers has at most one I-limit. If u, v are I-limit this sequence, then for every $\varepsilon > 0$

$$A\left(\frac{\varepsilon}{2}\right) = \{n \in N;\ D(u_n, u) \geq \varepsilon\} \in I, B\left(\frac{\varepsilon}{2}\right) = \{n \in N;\ D(u_n, v) \geq \varepsilon\} \in I,$$

$$C = A\left(\frac{\varepsilon}{2}\right) \cup B\left(\frac{\varepsilon}{2}\right) \in I,\ N - C \neq \emptyset.$$

For any $x \in N - C$

$$0 \leq D(u,v) \leq D(u,u_n) + D(u_n,v) < \varepsilon.$$

So $D(u,v) = 0$ and $u = v$.

A fuzzy number u is called the limit point of a sequence $\{u_n\}_{n=1}^{\infty}$ of fuzzy numbers, if u is limit of some its subsequence, i.e. $I_1 - \lim_{n\to\infty} u_{k_n} = u$, where $\{k_n\}_{n=1}^{\infty}$ is an increasing sequence of natural numbers.

Proposition 5. Let $I - \lim_{n\to\infty} u_n = u$. Then u is a limit point of the sequence $\{u_n\}_{n=1}^{\infty}$.

Proof. For any $n \in N$ the set $\left\{k \in N;\ D(u_k,u) < \frac{1}{n}\right\}$ is infinite. There is an increasing sequence of natural numbers $\{k_n\}_{n=1}^{\infty}$ such that $D(u_{k_n},u) < \frac{1}{n}$ for every $n \in N$. It is evident that $I_1 - \lim_{n\to\infty} u_{k_n} = u$. So u is a limit point of a sequence $\{u_n\}_{n=1}^{\infty}$.

Proposition 6. If $I - \lim_{n\to\infty} u_n = u$ and $I - \lim_{n\to\infty} v_n = v$, then

$$I - \lim_{n\to\infty}(u_n + v_n) = u + v.$$

Proof. Denote us

$$A(\varepsilon) = \{n \in N; D(u_n + v_n, u + v) \geq \varepsilon\},$$

$$A_1\left(\tfrac{\varepsilon}{2}\right) = \left\{n \in N; D(u_n,u) \geq \frac{\varepsilon}{2}\right\},\ A_2\left(\tfrac{\varepsilon}{2}\right) = \left\{n \in N; D(v_n,v) \geq \frac{\varepsilon}{2}\right\}$$

for any $\varepsilon > 0$. Obviously $A_1\left(\frac{\varepsilon}{2}\right), A_2\left(\frac{\varepsilon}{2}\right) \in I$.
For any $n \in N$ $D(u_n + v_n, u + v) \leq D(u_n,u) + D(v_n,v)$ (see (ii)).
The following implication is true

$$\left(n \notin A_1\left(\tfrac{\varepsilon}{2}\right) \wedge n \notin A_2\left(\tfrac{\varepsilon}{2}\right)\right) \Rightarrow n \notin A(\varepsilon),$$

or otherwise equivalently

$$n \in A(\varepsilon) \Rightarrow \left(n \in A_1\left(\frac{\varepsilon}{2}\right) \vee n \in A_2\left(\frac{\varepsilon}{2}\right)\right)$$

accordingly $A(\varepsilon) \subseteq A_1\left(\frac{\varepsilon}{2}\right) \cup A_2\left(\frac{\varepsilon}{2}\right) \in I$, consequently $A(\varepsilon) \in I$.

Proposition 7. If $I - \lim_{n\to\infty} u_n = u$ and $I - \lim_{n\to\infty} v_n = v$, then

$$I - \lim_{n\to\infty}(u_n v_n) = uv.$$

Proof. For any $\in N$ $D(u_n v_n, uv) \leq D(u_n v_n, u v_n) + D(u v_n, uv)$. By lemma 3 we obtain $K, L \in R^+$ such that $|u| \leq L$ and $|v_n| \leq K$ for every $n \in B \in \mathcal{F}(I)$. Let $\varepsilon > 0$. The sets $M_1 = \left\{n \in N; D(u_n,u) < \frac{\varepsilon}{2K}\right\}$, $M_2 = \left\{n \in N; D(v_n,v) < \frac{\varepsilon}{2L}\right\}$ belong to $\mathcal{F}(I)$. Evidently $M_1 \cap M_2 \cap B \in \mathcal{F}(I)$. So for any $n \in M_1 \cap M_2 \cap B$ we obtain

$$D(u_n v_n, uv) \leq D(Ku_n, Ku) + D(Lv_n, Lv) =$$

$$KD(u_n,u) + LD(v_n,v) < K \cdot \frac{\varepsilon}{2K} + L \cdot \frac{\varepsilon}{2L} = \varepsilon.$$

Hence $\{n \in N; D(u_n v_n, uv) \geq \varepsilon\} \in I$ and so $I - \lim_{n\to\infty}(u_n v_n) = uv$.

I-continuity of the fuzzy number-valued functions

It is possible to define I-convergence of sequences in any topological space. Let $(X,\mathcal{T})$ be a topology space with topology $\mathcal{T}$ and let I be an admissible ideal.

Definition 8. A sequence $\{x_n\}_{n=1}^{\infty}$ of points from X is said to converge to x with respect to the ideal I (we write $I-\lim_{n\to\infty} x_n = x$) if

$$A(T) = \{n \in N;\ x_n \notin T\} \in I$$

for any $T \in \mathcal{T}, x \in T$.

We will consider functions $f: X \to E$, where $(X,\mathcal{T})$ is a topology space and E is the set of fuzzy numbers.

Definition 9. Let I_1 and I_2 be admissible ideals. A function $f: X \to E$ is said to be (I_1, I_2)-continuous at x_0 , $x_0 \in X$, if

$$I_1 - \lim_{n\to\infty} x_n = x_0 \Longrightarrow I_2 - \lim_{n\to\infty} f(x_n) = f(x_0)$$

holds for every sequence $\{x_n\}_{n=1}^{\infty}$ of points from X.

Proposition 10. If functions f,g are (I_1, I_2)- continuous at x_0, then $f+g$ is (I_1, I_2)-continuous at x_0 .

Proof. Let $I_1 - \lim_{n\to\infty} x_n = x_0$, $A_{f+g}(\varepsilon) = \{n \in N; D((f+g)(x_n), (f+g)(x_0)) \geq \varepsilon\}$, $A_f = \left\{n \in N; D(f(x_n), f(x_0)) \geq \frac{\varepsilon}{2}\right\}$, $A_g = \left\{n \in N; D(g(x_n), g(x_0)) \geq \frac{\varepsilon}{2}\right\}$ for any $\varepsilon > 0$. For any $n \in N$

$$D((f+g)(x_n), (f+g)(x_0)) = D(f(x_n) + g(x_n), f(x_0) + g(x_0)) \leq$$
$$D(f(x_n), f(x_0)) + D(g(x_n), g(x_0)).$$

By proposition 5 $A_{f+g}(\varepsilon) \subseteq A_f \cup A_g \in I_2$, consequently $A(\varepsilon) \in I_2$.

Proposition 11. If functions f,g are (I_1, I_2)- continuous at x_0, then $f \cdot g$ is (I_1, I_2)-continuous at x_0 .

Proof. Let $I_1 - \lim_{n\to\infty} x_n = x_0$. By lemma 3 we obtain $K, L \in R^+$ such that $|f(x_{0)}| \leq L$ $|g(x_n)| \leq K$ for every $n \in B \in \mathcal{F}(I_2)$. Let $\varepsilon > 0$. The sets

$$M_1 = \left\{n \in N; D(f(x_n), f(x_0)) < \frac{\varepsilon}{2K}\right\}, M_2 = \left\{n \in N; D(g(x_n), g(x_0)) < \frac{\varepsilon}{2L}\right\}$$

belong to $\mathcal{F}(I_2)$. So for any $n \in M_1 \cap M_2 \cap B \in \mathcal{F}(I_2)$ we obtain

$$D((f\cdot g)(x_n), (f\cdot g)(x_0)) = D(f(x_n)g(x_n), f(x_0)g(x_0)) \leq$$
$$D(f(x_n)g(x_n), f(x_0)g(x_n)) + D(f(x_0)g(x_n), f(x_0)g(x_0)) \leq$$
$$K\, D(f(x_n), f(x_0)) + LD(g(x_n), g(x_0)) < \varepsilon.$$

Hence $\left\{n \in N; D\big(f(x_n)g(x_n), f(x_0)g(x_0)\big) \geq \varepsilon\right\} \in I_2$ and so
$I_2 - \lim_{n\to\infty} f(x_n)g(x_n) = f(x_0)g(x_0)$.

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Received on April 12, 2013.

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