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LIMIT THEOREMS FOR TOPOLOGICAL GROUP-VALUED MEASURES WITH RESPECT TO FILTER CONVERGENCE

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ABSTRACT. *Some Nikodým boundedness and limit theorems for topological group-valued measures are proved in the context of filter convergence.*

KEY WORDS: *topological group, filter, filter convergence, Brooks-Jewett theorem, Nikodým boundedness theorem, Stone Isomorphism Theorem.*

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1 Introduction

In this paper we give some versions of convergence and Nikodým boundedness theorems for topological group-valued measures with respect to filters. In this setting, in general it is impossible to obtain results analogous to the classical ones, even for positive real-valued measures (see for instance [2, Example 3.4], [4, Remark 3.8]). However, for suitable classes of filters/ideals, it is possible to get different versions of such kinds of theorems (see for instance [1] for real- valued measures and [2, 3, 4] for lattice group-valued measures).

Here, using some techniques similar to those in [5, 6, 7], we deal with the topological group setting. We first consider the σ-additive case and then, using the Stone Isomorphism Theorem, we investigate also the finitely additive case. We consider a concept of semivariation analogous to the classical one and we deal directly with measures defined on a σ-algebra of parts of an abstract set, without needing preliminary results for measures defined on the class of all subsets of $\mathbb{N}$, and giving an approach different from that in [3, 4].

2 Preliminaries

Let $Z \neq \emptyset$ be any set. A *filter* $\mathcal{F}$ of Z is a nonempty collection of subsets of Z with $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, and such that for each $A \in \mathcal{F}$ and $B \supset A$ we get $B \in \mathcal{F}$.

A filter of Z is said to be *free* iff it contains the filter $\mathcal{F}_{\text{cofin}}$ of all cofinite subsets of Z.

Let P be a countable set, and $\mathcal{F}$ be a filter of P. A subset of P is $\mathcal{F}$-*stationary* iff it has nonempty intersection with every element of $\mathcal{F}$. We denote by $\mathcal{F}^*$ the family of all $\mathcal{F}$-stationary subsets of P.

If $I \in \mathcal{F}^*$, then the *trace* $\mathcal{F}(I)$ of $\mathcal{F}$ on I is the family $\{A \cap I : A \in \mathcal{F}\}$.

A filter $\mathcal{F}$ of P is *diagonal* iff for every sequence $(A_n)_n$ in $\mathcal{F}$ and for each $I \in \mathcal{F}^*$ there exists a set $J \subset I$, $J \in \mathcal{F}^*$ such that the set $J \setminus A_n$ is finite for all $n \in \mathbb{N}$ (see also [3, 4]).

Observe that $\mathcal{F}(I)$ is a filter of I. Indeed, if F_1, $F_2 \in \mathcal{F}(I)$, then $(F_1 \cap F_2) \cap I = (F_1 \cap I) \cap (F_2 \cap I) \in \mathcal{F}$, and hence $F_1 \cap F_2 \in \mathcal{F}(I)$.

Let now $F^\circ \in \mathcal{F}$ and $F^\circ \cap I \subset F' \subset I$, and set $F^* := F' \cup F^\circ$: then $F^* \in \mathcal{F}$ and $F^* \cap I \supset F^\circ \cap I$. It is readily seen that $F' \subset F^* \cap I$. To prove the converse inclusion, observe that $F^* \cap I = (F' \cap I) \cup (F^\circ \cap I) \subset F'$. Hence, $F' = F^* \cap I$ belongs to $\mathcal{F}(I)$, and thus we get the claim.

Given an infinite set $I \subset P$, a *blocking* of I is a countable partition $\{D_k : k \in \mathbb{N}\}$ of I into nonempty finite subsets.

A filter $\mathcal{F}$ of P is said to be *block-respecting* iff for every $I \in \mathcal{F}^*$ and for each blocking $\{D_k : k \in \mathbb{N}\}$ of I there is a set $J \in \mathcal{F}^*$, $J \subset I$ with $\#(J \cap D_k) = 1$ for all $k \in \mathbb{N}$, where $\#$ denotes the number of elements of the set into brackets.
Some examples of filters satisfying these properties and of filters lacking them can be found in [1].

The following result will be useful in the sequel.

Proposition 2.1 *If* $\mathcal{F}$ *is a block-respecting filter of* $\mathbb{N}$*, then* $\mathcal{F}(I)$ *is a block-respecting filter of* I *for every* $I \in \mathcal{F}^*$.
Proof: Let $I \subset \mathbb{N}$ be any $\mathcal{F}$-stationary set, $L \subset I$ be any $\mathcal{F}(I)$-stationary set and $\{D_k : k \in \mathbb{N}\}$ be any blocking of L. If $F^\circ \in \mathcal{F}$, then $\emptyset \neq L \cap F^\circ \cap I = L \cap F^\circ$, and so $L \in \mathcal{F}^*$. By hypothesis, there exists a set $J \in \mathcal{F}^*$, $J \subset L$, with $\#(J \cap D_k) = 1$ for all $k \in \mathbb{N}$. In particular, $\emptyset \neq L \cap F^\circ = L \cap F^\circ \cap I$. From this it follows that $J \in \mathcal{F}(I)^*$. Thus we get the assertion. □

From now on $\mathcal{F}$ is a free filter of $\mathbb{N}$, $R = (R,+)$ is a Hausdorff complete abelian topological group satisfying the first axiom of countability, with neutral element 0, and $\mathcal{J}(0)$ denotes a basis of closed and symmetric neighborhoods of 0 (see also [5, 6, 7]). Moreover, given $k \in \mathbb{N}$ and $U_1, \ldots, U_k \subset R$, set $U_1 + \cdots + U_k := \{u_1 + \ldots + u_k : u_1 \in U_1, \ldots, u_k \in U_k\}$, and $kU := U + \cdots + U$ (k times).

We now give the notions of filter convergence and filter boundedness.

Let $(x_n)_n$ be a sequence in R and $x \in R$. We say that $\lim_n x_n = x$ iff for every $U \in \mathcal{J}(0)$ there is $n_0 \in \mathbb{N}$ with $x_n \in U$ for each $n \geq n_0$, and that $(\mathcal{F})\lim_n x_n = x$ iff $\{n \in \mathbb{N} : x_n - x \in U\} \in \mathcal{F}$ for every $U \in \mathcal{J}(0)$. Note that $\lim_n x_n = x$ iff $(\mathcal{F}_{\text{cofin}})\lim_n x_n = x$.

Let $(B_n)_n$ be a sequence of subsets of R. We say that $\lim_n B_n = 0$ iff for every $U \in \mathcal{J}(0)$ there is $n^* \in \mathbb{N}$ with $B_n \subset U$ for any $n \geq n^*$, and $(\mathcal{F})\lim_n B_n = 0$ iff the set $\{n \in \mathbb{N} : B_n \subset U\} \in \mathcal{F}$ for each $U \in \mathcal{J}(0)$. Observe that $\lim_n B_n = 0$ iff $(\mathcal{F}_{\text{cofin}})\lim_n B_n = 0$.

Let $(U_n)_n$ be an increasing sequence in $\mathcal{J}(0)$. A sequence $(x_n)_n$ in R is $\mathcal{F}$*-bounded by* $(U_n)_n$ iff $\{n \in \mathbb{N} : x_n \in U_n\} \in \mathcal{F}$. We say that $(x_n)_n$ is *eventually bounded by* $(U_n)_n$ iff it is $\mathcal{F}_{\text{cofin}}$-bounded by $(U_n)_n$.

From now on, $\mathcal{E}$ is a σ-algebra of subsets of an infinite set G. If $m : \mathcal{E} \to R$ be a finitely additive measure, set $m^+(A) = \{m(B) : B \in \mathcal{E}, B \subset A\}$, $A \in \mathcal{E}$. We say that m is (s)*-bounded* iff $\lim_n m^+(A_n) = 0$ for every disjoint sequence $(A_n)_n$ in $\mathcal{E}$, and that m is σ-

additive iff $\lim_n m^+(C_n)=0$ for every decreasing sequence $(C_n)_n$ in $\mathcal{E}$ with $\cap_{n=1}^{\infty} C_n=\emptyset$ (see also [5, 6, 7]).

We now prove the next technical lemma (see also [1, Lemma 3.3], [3, Lemma 2.2] and [4, Lemma 3.1]).

Lemma 2.2 *Let $(a_{i,n})_{i,n}$ be a double sequence in R, and $\mathcal{F}$ be a diagonal filter.*

α) *If $(\mathcal{F})\lim_{i\in\mathbb{N}} a_{i,n}=0$ for each $n\in\mathbb{N}$, then for every $I\in\mathcal{F}^*$ there exists $J\in\mathcal{F}^*$, $J\subset I$ such that $\lim_{i\in J} a_{i,n}=0$ for all $n\in\mathbb{N}$.*

$\alpha\alpha$) *If $(V_i)_i$ is an increasing sequence in $\mathcal{J}(0)$ and $(a_{i,n})_i$ is $\mathcal{F}$-bounded by $(V_i)_i$ for every $n\in\mathbb{N}$, then for each $I\in\mathcal{F}^*$ there is $J\in\mathcal{F}^*$, $J\subset I$ such that $(a_{i,n})_i$ is eventually bounded by $(V_i)_i$.*

Proof: α) Let $(U_p)_p$ be a countable basis of neighborhoods of 0. By hypothesis, for every n, $p\in\mathbb{N}$ we have $A_{n,p}:=\{i\in\mathbb{N}: a_{i,n}\in U_p\}\in\mathcal{F}$. Since $\mathcal{F}$ is diagonal, for each $I\in\mathcal{F}^*$ there is $J\in\mathcal{F}^*$, $J\subset I$, such that for every n, $p\in\mathbb{N}$ the set $J\setminus A_{n,p}$ is finite. Thus, for every n, $p\in\mathbb{N}$ there is $\bar{i}\in\mathbb{N}$ (without loss of generality $\bar{i}\in J$) with $a_{i,n}\in U_p$ for all $i\geq\bar{i}$, $i\in J$. This proves α).

The proof of $\alpha\alpha$) is analogous, taking the sets $A_n^*:=\{i\in\mathbb{N}: a_{i,n}\in V_i\}$, $n\in\mathbb{N}$, instead of the $A_{n,p}$'s. □

3 The main results

We begin with a convergence theorem for topological group-valued measures (for related results see also [1, Theorems 2.6 and 3.5] for the Banach space setting and [3, Lemma 3.1 and Theorems 3.1, 4.1 and 4.2] for the lattice group context). Note that the hypothesis that the involved filter is block- respecting is essential, even when $R=\mathbb{R}$ (see also [1, Remark 3.4]).

Theorem 3.1 *Let $\mathcal{F}$ be a block-respecting filter of $\mathbb{N}$, $m_j:\mathcal{E}\to R$, $j\in\mathbb{N}$, be a sequence of σ-additive measures, $(A_n)_n$ be a disjoint sequence in $\mathcal{E}$, with*

i) $\lim_j m_j(A_n)=0$ *for any $n\in\mathbb{N}$, and*

ii) $(\mathcal{F})\lim_j m_j(\cup_{p\in P} A_p)=0$ *for every $P\subset\mathbb{N}$. Then,*

β) *for every strictly increasing sequence $(l_n)_n$ in $\mathbb{N}$ we get*

$$(\mathcal{F})\lim_n m_n(A_{l_n})=0; \tag{1}$$

$\beta\beta$) *if $\mathcal{F}$ is also diagonal, then the only condition ii) is sufficient to get (1).*

Proof: Put $H_n:=A_{l_n}$, $n\in\mathbb{N}$. If we deny the thesis, then there is $U\in\mathcal{J}(0)$ with $C:=\{n\in\mathbb{N}: m_n(H_n)\in U\}\notin\mathcal{F}$. Note that $I:=\mathbb{N}\setminus C=\{n\in\mathbb{N}: m_n(H_n)\notin U\}\in\mathcal{F}^*$: otherwise there is $F'\in\mathcal{F}$ with $I\cap F'=\emptyset$, namely $F'\subset C$ and hence, $C\in\mathcal{F}$, a contradiction.

Let now $(U_k)_k$ be a decreasing sequence in $\mathcal{J}(0)$, with $U_0=U$, and $2U_k\subset U_{k-1}$ for every $k\in\mathbb{N}$ (see also [6]). It is not difficult to see that $lU_k\subset U_{k-l+1}$ for all k, $l\in\mathbb{N}$ with $l\leq k+1$.

Let $N_0 = 1$. By σ-additivity of m_1, there exists a cofinite subset $P_1 \subset \mathbb{N}$, with $N_0 < p_1 := \min P_1$, and $m_1^+(F_1) \subset U_1$, where $F_1 := \cup_{t \in P_1} H_t$. By i), there is an integer $N_1 > p_1$ with $m_i(H_t) \in U_1$ whenever $i \geq N_1$ and $t = 1, \ldots, p_1$.
By σ-additivity of m_1, $m_2, \ldots, m_{N_1}$, there is a cofinite subset $P_2 \subset P_1$, with $N_1 < p_2 := \min P_2$, and $m_r^+(F_2) \subset U_2$ for every $r = 1, \ldots, N_1$, where $F_2 := \cup_{t \in P_2} H_t$. Arguing as above, there exists $N_2 > p_2$ with $m_i(H_t) \in U_2$ whenever $i \geq N_2$ and $t = 1, \ldots, p_2$.
Proceeding by induction, we find: a strictly decreasing sequence $(P_k)_k$ of cofinite subsets of $\mathbb{N}$, a strictly decreasing sequence $(F_k)_k$ in $\mathcal{E}$ and two strictly increasing sequences $(N_k)_k$, $(p_k)_k$ in $\mathbb{N}$ such that, for every $k \in \mathbb{N}$,
3.1.1) $N_k > p_k$, $p_{k+1} > N_k$, $p_k = \min P_k$; $F_k = \cup_{t \in P_k} H_t$;
3.1.2) $m_r^+(F_{k+1}) \subset U_{k+1}$ for all $r = 1, \ldots, N_k$;
3.1.3) $m_i(H_t) \in U_k$ whenever $i \geq N_k$ and $t = 1, \ldots, p_k$.
Since $\mathcal{F}$ is block-respecting, there is $J := \{j_1, j_2, \ldots\} \in \mathcal{F}^*$, $J \subset I$, with $N_k \leq j_k < N_{k+1}$ for every $k \in \mathbb{N}$. As $J \in \mathcal{F}^*$, then either $J_1 := \{j_1, j_3, j_5, \ldots\} \in \mathcal{F}^*$ or $J_2 := \{j_2, j_4, j_6, \ldots\} \in \mathcal{F}^*$. Without loss of generality, let $J_1 \in \mathcal{F}^*$ (see also [1, 3, 4]). Put $A := \cup_{h=1}^{\infty} H_{j_{2h-1}}$. We get:

$$m_{j_1}(A) = m_{j_1}(H_{j_1}) + m_{j_1}(H_{j_3} \cup H_{j_5} \cup \ldots);$$
$$m_{j_{2h-1}}(A) = m_{j_{2h-1}}(H_{j_1} \cup H_{j_3} \cup \ldots \cup H_{j_{2h-3}}) +$$
$$+ m_{j_{2h-1}}(H_{j_{2h-1}}) + m_{j_{2h-1}}(H_{j_{2h+1}} \cup H_{j_{2h+3}} \cup \ldots), \quad h \geq 2.$$

Since $j_{2h-1} < N_{2h-1} < p_{2h}$ and

$$H_{j_{2h+1}} \cup H_{j_{2h+3}} \cup \ldots \subset \cup_{l=p_{2h+1}}^{\infty} H_l = F_{2h+1} \quad \text{for every } h \in \mathbb{N},$$

from (3) and 3.1.2) used with $k = 2h$ we obtain

$$m_{j_{2h-1}}(H_{j_{2h+1}} \cup H_{j_{2h+3}} \cup \ldots) \in U_{2h+1} \subset U_3.$$

Moreover, since $j_{2h-3} < N_{2h-3} < p_{2h-2} < p_{2h-1}$ for every $h \geq 2$, from 3.1.3) used with $k = 2h-1$ we get $m_{j_{2h-1}}(H_{j_l}) \in U_{2h-1}$, $h \geq 2$, $l = 1, 3, \ldots, 2h-3$, and hence

$$m_{j_{2h-1}}(H_{j_1} \cup H_{j_3} \cup \ldots \cup H_{j_{2h-3}}) \in (h-1)U_{2h-1} \subset U_h \subset U_3.$$

If $m_{j_{2h-1}}(A) \in U_1$, then from (2), (4) and (5) we have $m_{j_1}(H_{j_1}) \in U_1 + U_2 \subset U$ and $m_{j_{2h-1}}(H_{j_{2h-1}}) \in U_1 + U_2 + U_3 \subset U_1 + U_1 \subset U$ for all $h \geq 2$. But we know that $m_{j_{2h-1}}(H_{j_{2h-1}}) \notin U$, and so we have a contradiction. Thus, we get that $m_{j_{2h-1}}(A) \notin U_1$ for all $h \in \mathbb{N}$, and so $L := \{l \in \mathbb{N} : m_l(A) \notin U\} \in \mathcal{F}^*$. Since, by ii), $\mathbb{N} \setminus L \in \mathcal{F}$, we obtain $L \cap (\mathbb{N} \setminus L) \neq \emptyset$, which is absurd. This proves $\beta)$.
We now prove $\beta\beta)$. If we deny the thesis, then, proceeding analogously as in the proof of $\beta)$, we find $I \in \mathcal{F}^*$ and $U \in \mathcal{J}(0)$ with $m_n(A_{l_n}) \notin U$ for each $n \in I$. By Lemma 2.2, there is $J \in \mathcal{F}^*$, $J \subset I$, with $\lim_{j \in J} m_j(A_{l_n}) = 0$ for any $n \in \mathbb{N}$. Note that the sequence $m_n(A_{l_n})$, $n \in \mathbb{N}$, does not $(\mathcal{F}(J))$-converge to 0 (see also [1]). Since $J \in \mathcal{F}^*$ and $\mathcal{F}$ is block-

respecting, then, by Proposition 2.1, $\mathsf{F}(J)$ is block-respecting too. As $\mathsf{F}(J) \supset \mathsf{F}$, it is easy to see that $(A_{l_n})_n$ satisfies ii) with respect to $\mathsf{F}(J)$. By $\beta)$ used with $\mathsf{F}(J)$ and $(A_{l_n})_n$, it follows that $(\mathsf{F}(J))\lim_n m_n(A_{l_n}) = 0$, obtaining a contradiction. This proves $\beta\beta)$.

We now extend Theorem 3.1 to the setting of finitely additive measures.

Theorem 3.2 *Let $(A_n)_n$ be as in Theorem 3.1, F be a block-respecting filter of N, $m_j : \mathsf{E} \to R$, $j \in \mathsf{N}$, be a sequence of finitely additive s-bounded measures, and assume that*

i) $\lim_j m_j(A_n) = 0$ for any $n \in \mathsf{N}$;

ii) $(\mathsf{F})\lim_j \sum_{p\in P} m_j(A_p) = 0$ for every $P \subset \mathsf{N}$.

Then for every strictly increasing sequence $(l_n)_n$ in N we get

$$(\mathsf{F})\lim_n m_n(A_{l_n}) = 0.$$

If F is also diagonal, then the only condition ii) is enough to get (6).

Proof: By the Stone Isomorphism Theorem (see also [8]) there is a topological space Ω, such that E is isomorphic to the algebra Q of all clopen subsets of Ω. Let us denote by $\psi : \mathsf{E} \to \mathsf{Q}$ such an isomorphism, and let $\Sigma(\mathsf{Q})$ be the σ-algebra generated by Q. Thus for every $j \in \mathsf{N}$ the measure $m_j \circ \psi^{-1} : \mathsf{Q} \to R$ is σ-additive and admits a σ-additive extension $\mu_j : \Sigma(\mathsf{Q}) \to R$ (see also [5, 9, 10]), satisfying together with the sets $\psi^{-1}(A_n)$, $n \in \mathsf{N}$, the conditions i) and ii) of Theorem 3.1. Hence, $0 = (\mathsf{F})\lim_n \mu_n(\psi^{-1}(A_{l_n})) = (\mathsf{F})\lim_n m_n(A_{l_n})$, and so we get (6).

The last assertion follows by arguing as in the proof of Theorem 3.1, $\beta\beta)$.

We now give a version of the Nikodým boundedness theorem for topological group-valued measures (for the Riesz space context, see also [4, Lemma 3.4 and Theorem 3.5]).

Theorem 3.3 *Let F be a block respecting filter of N, $m_j : \mathsf{E} \to R$, $j \in \mathsf{N}$, be a sequence of finitely additive (s)-bounded measures, and $(A_n)_n$ be a disjoint sequence in E. Let $U \in \mathsf{J}(0)$, $(W_n)_n$ be an increasing sequence in $\mathsf{J}(0)$, and set $V_n := nW_n + U$, $n \in \mathsf{N}$. Suppose that:*

j) the set $\{m_n(A_p) : n \in \mathsf{N}\}$ is eventually bounded by $(W_n)_n$ for each $p \in \mathsf{N}$;

jj) the set $\{\sum_{p\in P} m_j(A_p) : n \in \mathsf{N}\}$ is F-bounded by $(W_n)_n$ for any $p \in \mathsf{N}$. Then

$\gamma)$ for every strictly increasing sequence $(l_n)_n$ in N, the set $D := \{m_n(A_{l_n}) : n \in \mathsf{N}\}$ is F-bounded by $(V_n)_n$.

$\gamma\gamma)$ If F is also diagonal, then the only condition jj) is enough in order that D is F-bounded by $(V_n)_n$.

Proof: For every $n \in \mathsf{N}$, let $H_n := A_{l_n}$. First of all note that, if the m_j's are σ-additive, then the proof of $\gamma)$ is similar to that of Theorem 3.1, $\beta)$. Indeed, if the thesis of the theorem is not true, then $I := \{n \in \mathsf{N} : m_n(H_n) \notin V_n\} \in \mathsf{F}^*$. By σ-additivity of m_1, there is a cofinite set $P_1 \subset \mathbb{N}$, with $1 < p_1 = \min P_1$ and $m_1^+(F_1) \subset U$, where $F_1 := \cup_{t \in P_1} H_t$. By j) there is $N_1 > p_1$ with $m_i(H_t) \in W_i$ for each $i \geq N_1$ and $t = 1, \ldots, N_1$. By induction, there are a strictly decreasing sequence $(F_k)_k$ in $\mathcal{E}$ and two strictly increasing sequences $(N_k)_k$, $(p_k)_k$ in $\mathbb{N}$ such that, for each $k \in \mathbb{N}$,

3.3.1) $N_k > p_k$, $p_{k+1} > N_k$; $m_r^+(F_{k+1}) \subset U_{k+1}$ for every $r = 1, \ldots, N_k$;

3.3.2) $m_i(H_t) \in W_i$ for any $i \geq N_k$ and $t = 1, \ldots, p_k$.

As $\mathcal{F}$ is block-respecting, we find a set $J_1 := \{j_1, j_3, j_5, \ldots\} \in \mathcal{F}^*$, $J_1 \subset I$, with $N_k \leq j_k < N_{k+1}$ for every $k \in \mathbb{N}$. For any $h \in \mathbb{N}$ we have:

$$m_{j_{2h-1}}(H_{j_{2h+1}} \cup H_{j_{2h+3}} \cup \ldots) \in U;$$

$m_{j_{2h-1}}(H_{j_l}) \in W_{2h-1}$, $h \geq 2$, $l = 1, 3, \ldots, 2h-3$, and

$$m_{j_{2h-1}}(H_{j_1} \cup H_{j_3} \cup \ldots \cup H_{j_{2h-3}}) \in (h-1) W_{2h-1}.$$

Let now $A := \cup_{h=1}^{\infty} H_{j_{2h-1}}$. If $m_{j_{2h-1}}(A) \in W_{j_{2h-1}}$, then from (2), (7) and (8) we obtain

$$m_{j_{2h-1}}(A_{j_{2h-1}}) \in h W_{2h-1} + U \subset j_{2h-1} W_{j_{2h-1}} + U = V_{j_{2h-1}}$$

and

$$m_{j_1}(A_{j_1}) \in W_{j_1} + U \subset j_1 W_{j_1} + U = V_{j_1}.$$

This contradicts the fact that $m_{j_{2h-1}}(H_{j_{2h-1}}) \notin V_{j_{2h-1}}$. Thus $m_{j_{2h-1}}(A) \notin W_{j_{2h-1}}$ for all $h \in \mathbb{N}$, and hence $\{l \in \mathbb{N} : m_l(A) \notin W_l\} \in \mathcal{F}^*$. From this, arguing as at the end of the proof of Theorem 3.1, $\beta)$, we get a contradiction, and this proves $\gamma)$. From $\gamma)$, proceeding as in the proof of Theorem 3.1, $\beta\beta)$, we get $\gamma\gamma)$, at least in the σ-additive case.

When the m_j's are finitely additive and (s)-bounded, it is enough to use the results obtained in the σ-additive setting and to argue as in Theorem 3.2.

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