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ABOUT CONFERENCE

12-th mathematical conference in Nitra is conference with more than 10 years of tradition which creates a forum for discussion of issues in mathematical education and selected topics in pure and applied mathematics. This conference aims to bring together the educational scientists, education experts, teachers, graduate students and civil society organization and representatives to share and to discuss theoretical and empirical knowledge in these branches:

- Innovations and research in education of mathematics teachers at all school levels;
- Current problems and trends in mathematics education;
- Assessment and evaluation in mathematics education;
- ICT in mathematics education;
- Psychology in mathematics education;
- Research in mathematics education.

INVITED LECTURERS

Assoc. Prof. RNDr. Ivan Kalaš, PhD., Department of Informatics Education, Comenius University, Bratislava, Slovak Republic

Ivan Kalas is a professor of Informatics Education. For more than 20 years, he concentrates on developing Informatics (Computing) curricula for preschool, primary and secondary stages, developing textbooks and other teaching/learning materials for Informatics and ICT in education. Professor Kalas is also interested in strategies for developing digital literacy of future and in-service teachers and enhancing learning processes through digital technologies. Professor Kalas works at the Department of Informatics Education, Comenius University, Bratislava where he leads educational research and doctoral school in the field of Technology Enhanced Learning. Professor Kalas is co-author of several educational software environments for children, which have dozens of localizations throughout the world and are being used in thousands of schools. Since 2008, professor Kalas is a member of the International Advisory Board of the Microsoft Partners in Learning programme. Since 2009, he is a member of the Governing Board of the UNESCO Institute for Information Technologies in Education. In 2010, professor Kalas conducted an analytical study for UNESCO titled Recognizing the potential of ICT in early childhood education. Since 2013, he is a Visiting Professor of the Institute of Education, University of London.

Lecture: TEACHERS CAUGHT IN THE DIGITAL NET

Annotation: I will pursue the topic, which is rather annoying – sometimes discussed too much, sometimes ignored, but always crippled by education policy makers and misunderstood by media: I will talk about understanding and discovering the potential of digital technologies to support the learning processes and complex development of our children (in particular, in kindergartens, primary, and secondary schools).

Productive integration of these technologies to support learning (on purpose I did not say teaching) does not advance as smoothly and quickly as we believed (and promised to everybody) some ten or twenty years ago. Is it good? Is it wrong? Did we overestimate the importance of new technologies? Did we fail in convincing others of their potential? Or did we completely miss the point?

I will ponder on the reasons, which obstruct and retard more vigorous exploitation of that potential. I will comment on actual trends, successes and failures in some other countries. I will also briefly present some of the projects I am currently involved in. My plan is to fiddle with one or two provocative ideas and give time to discuss them with the audience.

Assoc. Prof. Iveta Scholtzová, PhD., Faculty of Education University of Prešov in Prešov, Prešov, Slovak Republic

Iveta Scholtzová teaches mathematical disciplines in the field of Preschool and Elementary Education, Bachelor's degree program of Preschool and Elementary Education and Master's degree program of Teaching in the Primary Education. Her research interests include: comparative study of primary mathematics in Slovakia and abroad, philosophical and curricular transformation of mathematical training of pre-elementary and elementary teachers, cognitive aspects of mathematical education, incorporation of combinatorics, probability and statistics to primary education.

Lecture: DETERMINANTS OF PRIMARY MATHEMATICS EDUCATION - A NATIONAL AND INTERNATIONAL CONTEXT

Annotation: The results of the international surveys TIMSS (Trends in International Mathematics and Science Study) and PISA (Programme for International Student Assessment) indicate average or even below average performance of Slovak students in mathematics or numeracy compared with the OECD countries and other partner countries. Curricular transformation of primary education bring about some open issues and challenges also for primary mathematics education. The ever present question is: who, what, how and why determines the mathematical education on the primary level. To find the answers, it is necessary to analyse the existing situation in Slovakia in comparison with the data obtained from abroad.

Assoc. Prof. RNDr. Mária Kmeťová, PhD., Faculty of Natural Sciences Constantine the Philosopher University in Nitra, Nitra, Slovak Republic

Mária Kmeťová is an Associate Professor in the Department of Mathematics of Faculty of Natural Sciences Constantine the Philosopher University in Nitra. She teaches geometrical disciplines as Analytical and Constructive Geometry, Computer Geometry and Geometric Modelling for students of mathematics teaching and applied informatics. Her research interests are Mathematics Education, Geometry and Computer Graphics.

Lecture: FROM PROJECTIVE GEOMETRY TO COMPUTER GRAPHICS

Annotation: In the lecture we will study the segment of geometry dealing with concept of infinity, the origin of the projective geometry and follow-through the way which leads to geometric modelling of special curves and surfaces in computer graphics.

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FACULTY OF NATURAL SCIENCES
CONSTANTINE THE PHILOSOPHER UNIVERSITY IN NITRA
ACTA MATHEMATICA 17

FROM PROJECTIVE GEOMETRY TO COMPUTER GRAPHICS

MÁRIA KMEŤOVÁ

ABSTRACT. *In the lecture we will study the segment of geometry dealing with concept of infinity, the origin of the projective geometry and follow-through the way which leads to geometric modelling of special curves and surfaces in computer graphics.*

KEY WORDS: *point at infinity, projective geometry, curves and surfaces in CAGD*

CLASSIFICATION: *G10, G90*

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Introduction

For study geometric objects for computer graphics and modelling is necessary some knowledge of projective geometry. However the geometry of real objects is Euclidean, the geometry of imaging an object is projective. Hence the study of computer graphics naturally involves both geometries [7]. Projective geometry is useful in two levels: one is such architects use projective geometry when drawing a building as it would appear to an observer. Computer graphic use it for modelling realistic scenes. The second level of use is projective geometry settings for the theory of modelling curves and surfaces, because many of their intrinsic properties are naturally understood in a projective context.

What is projective geometry?

The roots of projective geometry go back to the middle ages. It was in 1425 that the Italian architect Filippo Brunelleschi began to discuss the geometrical theory of perspective, which was consolidated into a treatise a few years later [1]. The concept of perspective occurs besides architecture in painting and astronomy, too.

If we compare the paintings of the Renaissance painters with the painters of the preceding period, we notice a difference in depiction of depth. The figures on Gothic pictures are placed beside one another without any attempt to capture the depth of the space. It was the advent of the renaissance that led artists to study the techniques necessary for realistic rendering. The Renaissance painters wanted to paint the world as they saw it, to paint it from a particular point of view, to paint in perspective, to evoke the illusion of depth [4, 15]. Albrecht Dürer (1471 – 1528) has several works (woodcuts), where he shows a method for create a perspective 2-dimensional map of the 3-dimensional object. One of them is the woodcut "*A man drawing a lute*" (Nürnberg, 1525). This figure shows the scientific approach that Dürer took in order to master perspective: he used wires to record the perceived position of points that were marked on an object [4]. Central projection, illustrated on Dürer's pictures, forms the fundamental idea of projective geometry.

In central projection is a correspondence among points of the object and points of the image, which is established by associating to each point of the object the point of intersection of the image plane with the line containing the object point and the eye. For example, a pair of railroad tracks that disappear off into the distance, an artist adds a vanishing point to the picture. A vanishing point, in general, is that point in a picture at which two parallel lines in the scene appear to meet [17]. Then if we add a point at infinity, or ideal point of mentioned railroad tracks (and similarly for any direction of the object plane, and simultaneously in the image plane) the correspondence between the object and image planes becomes a one-to-one correspondence between the object plane completed with its ideal points and the image plane completed with its ideal points.

The important concept of point at infinity occurred independently to the German astronomer Johann Kepler (1571-1630) and the French architect Girard Desargues (1591-1661). Kepler (in his *Paralipomena in Vitellionem*, 1604) declared that a parabola has two foci, one of which is infinitely distant in both of two opposite directions, and that any point on the curve is joined to this “blind focus” by a line parallel to the axis. Desargues (in his *Brouillon project ...,* 1639) declared that parallel lines have a common end at an infinite distance, and again, “When no point of a line is at a finite distance, the line itself is at an infinite distance” [1]. Then the groundwork was laid to derive projective space from ordinary space by adding a common point at infinity for all lines parallel to each other and adding a common line at infinity for all planes to a given plane. Jean Victor Poncelet (1788-1867) fought in Napoleon’s Russian campaign (1812) until the Russians took him prisoner. As a prisoner at Saratoff on the Volga (1812-1814) he still had the vigour of spirit to implement a great work, he decided to reconstruct the whole science of geometry. The result was his epoch-making “*Traité des propriétés projectives des figures*“, published eight years later, in 1822 [2]. In this work was first made prominent the power of central projection. His leading idea was the study of projective properties, and as a foundation principle he introduced the anharmonic ratio (today known as cross-ratio) [16]. The discovery of the principle of duality was also claimed by Poncelet. This principle of geometric reciprocation has been greatly elaborated and has found its way into modern algebra and elementary geometry [16].

Our list of basic ideas of projective geometry is not exhaustive. We have just mentioned the fundamental ideas, which we need it as a tool for the description of rational curves and surfaces.

Early history of curves and surfaces

The earliest recorded use of curves in a manufacturing environment seems to go back to early AD Roman times, for the purpose of shipbuilding [5]. The vessel’s basic geometry has not changed for a long time. These techniques were perfected by the Venetians from the 13th to the 16th century. No drawings existed to define a ship hull; these became popular in England in the 1600s. The classical “spline”, a wooden beam which is used to draw smooth curves, was probably invented then. The earliest available mention of a “spline" seems to be from 1752 [5].

Another key event originated in aeronautics, where classical drafting methods were combined with computational techniques for the first time.

Some other early influential development for curves and surfaces was the advent of numerical control in the 1950s. In the U.S., General Motors used first CAD (Computer Aided Design) system developed by C. de Boor and W. Gordon. M. Sabin had key role in developing the CAD system for British Aircraft Corporation. He received his PhD from the Hungarian Academy of Sciences in 1977. Sabin developed many algorithms that were later “reinvented” [5].

A new concept

In 1959, the French car company Citroen hired a young mathematician Paul de Faget de Casteljau, who had just finished his PhD. He began to develop a system for design of curves and surfaces with using of Bernstein polynomials. The breakthrough insight was to use control polygons, a technique that was newer used before. De Casteljau's work was kept secret by Citroen for a long time.

During the early 1960s, Pierre Bézier headed the design department at Rénault, the competitor of Citroen, also located in Paris. Bézier's idea was to represent a basic curve as the intersection of two elliptic cylinders placed inside a parallelepiped [6]. Affine transformation of this parallelepiped would result the desired change of the curve (on affine map of the curve). Later, when Bézier used polynomial formulations of the initial concept, the result turned out to be identical to de Casteljau's curves; only the mathematics involved was different [5].

The de Casteljau algorithm for Bézier curves

The de Casteljau algorithm is the most fundamental algorithm in curve and surface modelling, but it is surprisingly simple. It is the beautiful interplay between geometry and algebra. A very intuitive geometric construction leads to a powerful theory [6].

Let us start with the *four tangent theorem* for conic in projective plane [3, 4, 11]. If one tangent is a line at infinity, we get the *three tangent theorem* for parabola [3] in affine plane: Let t_1, t_2, t_3 be three tangents of a parabola in tangent points V_0, V_0^2, V_2, respectively. Let the tangents at V_0 and V_2 intersect in V_1. Let the tangent at V_0^2 intersects the remaining tangents in V_0^1 and V_1^1 (Figure 1).

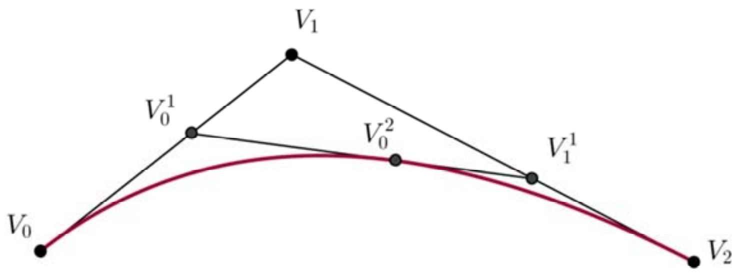

Figure 1

Then the following ratios are equal $(V_0V_1V_0^1) = (V_1V_2V_1^1) = (V_0^1V_1^1V_0^2)$.
It implies that

$$V_0^1 = (1-t)V_0 + tV_1,\ \ V_1^1 = (1-t)V_1 + tV_2 \text{ and } V_0^2 = (1-t)V_0^1 + tV_1^1,\ \ t\epsilon\langle 0,1\rangle.$$

Then after calculation we have the point of the parabola given as a barycentric combination of the points V_0, V_1, V_2.

$$V_0^2 = (1-t)^2V_0 + 2(1-t)tV_1 + t^2V_2$$

This is the simplest approach to the essential idea of de Casteljau algorithm. Point V_0^2, the result of the algorithm, is the point of the quadratic Bézier curve given as a linear combination of quadratic Bernstein polynomials. Generalization for degree n gives that a point $X(t)$ on the Bézier curve is given as

$$X(t) = \sum_{i=0}^{n} V_iB_i^n(t),$$

where $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ are Bernstein polynomials of degree n and $t\epsilon\langle 0,1\rangle$. The initial points $V_0, \dots, V_n$ are the so-called control points of the curve.

Rational Bézier curves

Projective geometry approach allows us to consider some control points at infinity (appear as "control vectors") [4]. Then we get a new type of curve with new possibilities of design it.
Another approach is to define Bézier curve $X(t) = \sum_{i=0}^{n} \mathbf{V_i}\, B_i^n(t)$ in projective space with control points $\boldsymbol{V_i} = [w_i V_i, w_i]$ in P^3, and to map it into the embedded affine space. In the analytic expression of the curve it yields that

$$X(t) = \frac{\sum_{i=0}^{n} w_i V_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}, \qquad t\epsilon\langle 0,1\rangle,$$

where V_i are control points in affine space and w_i are the so-called weights of the curve. This type of curve is known as a rational Bézier curve, because contains the ratio of two polynomials. Rational Bézier curves may be evaluated equally, using the concept of cross ratios, the fundamental invariant of projective geometry. Application of Menelaus` theorem leads to the same result [10, 11].
Rational curves have several advantages over polynomial Bézier curves. A degree two polynomial Bézier curve can only represent a parabola. Exact representation of circles and all conic sections requires rational degree two Bézier curves. The shape of the curve can be influenced not only with the shape of control polygon, but also with appropriate weights. A perspective projection of a Bézier curve is a rational Bézier curve.

Duality

A basic concept of projective geometry, the duality concept is widely used in curve modelling [13, 18]. Dual counterpart of plane Bézier curve defined by control points is a Bézier curve defined by control lines. The curve is thus given as the envelope of its tangents. In 3D space the dual curve is defined by control planes. The importance of this dual concept is in the theory of developable surface. The key observation is, that while the planes $\chi(t)$ are osculating planes of a curve, they themselves, being a one parameter family of planes, envelope a surface [4]. Such surfaces are called developable.

Triangular Bézier patches

In surface modelling, the finite piece of surface is called a patch. Two basic types are tensor product patches and triangular Bézier patches (Bézier triangles). When de Casteljau invented Bézier curves in 1959, he realized the need for the extension of the curve ideas for surfaces. The first surface type that he considered was what we now call Bézier triangles. This historical first of triangular patches is reflected by the mathematical statement that they are more natural generalization of Bézier curves than are tensor product patches [6]. Thus Bézier triangles can be perceived as a generalization of Bézier curves (for triangular domain instead of unit interval used for curves). Let a parameter $U = (u,v,w)$ be an element of triangular domain, where $0 \le u, v, w \le 1$ are barycentric coordinates. Expression of the Bézier triangle is then very similar to expression of the Bézier curve:

$$X(U) = \sum_{i+j+k=n} V_{ijk} B_{ijk}^n(U),$$

where V_{ijk} are control points and $B_{ijk}^n(U)$ are trivariate Bernstein polynomials

$$B_{ijk}^{n}(U) = B_{ijk}^{n}(u,v,w) = \frac{n!}{i!\,j!\,k!} u^{i} v^{j} w^{k}$$

($i, j, k \in \{0, 1, \dots, n\}$ and all subscripts sum to n).
Consider now a projective Bézier triangle; the previous polynomial Bézier triangle defined in projective space. Following the familiar theme of generating rational curve, we define a rational Bézier triangle as the projection of polynomial Bézier triangle to affine space.

$$X(U) = \frac{\sum_{i+j+k=n} w_{ijk} V_{ijk} B_{ijk}^{n}}{\sum_{i+j+k=n} w_{ijk} B_{ijk}^{n}},$$

where w_{ijk} are weights associated with the control points V_{ijk}, describes the rational Bézier triangle.

Quadrics

While the quadratic polynomial Bézier triangle represents a part of paraboloid (if it fulfils some extra condition [6, 8, 9,14]), their rational counterpart allows us to represent a part of quadric (hyperboloid, ellipsoid or especially sphere), because every quadric surface may be defined as a projective image of a paraboloid.
We get the following characterization for quadratic rational Bézier triangle lying on quadric surface [6, 8, 14]: A rational quadratic Bézier triangle is a part of quadric surface if and only if extensions of all tree their boundary curves meet in a common point of the quadric and have coplanar tangents there.
Using the previous condition and the so-called Patchwork Theorem [9] for degenerated 4-sided patches, it is possible to cover a sphere with combination of these patches [14]. (Figure 2 shows the set of triangular patches fulfilling the previous condition on the sphere, drawn with program Maple.)

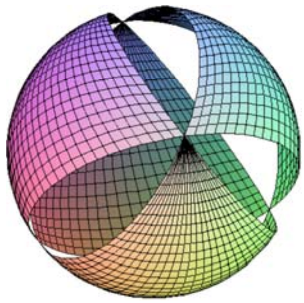

Figure 2

Summary

Projective geometry is a natural setting for many types of curves and surfaces used in computer aided geometric design (CAGD). The aim of this paper was to show the way from the beginnings of the projective geometry to its using in CAGD nowadays. Naturally, the overview is not complete, very important curves (e.g. NURBS, B-splines) and methods (e.g. stereographic projection [7, 18], WRD-construction [12]) and many others were not mentioned.

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DETERMINANTS OF PRIMARY MATHEMATICS EDUCATION – A NATIONAL AND INTERNATIONAL CONTEXT

IVETA SCHOLTZOVÁ

ABSTRACT. *The results of the international survey TIMSS indicate average or even below average performance of Slovak students in mathematics compared with the OECD countries and other partner countries. What are the factors that determine the mathematical education in primary school? During the last twenty years, teaching mathematics in Slovakia was affected by the two curricular modifications. A new curriculum for the year one of primary school was introduced in 1995 followed by the State Program Education for primary stage in 2008. They brought, on the one hand, a certain reduction in the scope of traditional mathematical curriculum in the first four years of primary school but, on the other hand, the mathematical instruction was enriched by some new topics. The teacher is an equally important factor, since he is responsible for implementation of mathematical education in school. Undergraduate training of prospective teachers - elementarists - builds on the mathematical training which they received in previous stages of education. In this view, it is important to consider the type of secondary school at which a teacher trainee had studied. Comparative analysis of the situation in Slovakia and abroad can yield new ideas to enhance the mathematical education.*

KEY WORDS: *Mathematics. Primary Education. Curriculum. Teacher.*

CLASSIFICATION: *B50, B70, D10*

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Content of Mathematical Education in Primary Stage in Slovakia

There have been no significant changes in the content of primary mathematics over the years in Slovakia- it appears so at first sight. After closer analysis, however, a different conclusion is drawn. The range of mathematical knowledge that the pupil should acquire at the end of the fourth year of primary school has been since 1989 reduced but at the same time enriched with some new portions of knowledge. (More details in [8].)

The period since 1989 can be divided into three stages:

- before 1995;
- from 1995, when the new curriculum for the first year of primary school entered into force, to 2008;
- from 2008, when State Program of Education for primary stage was introduce, up to the present.

Mathematics has been in terms of the number of hours per week delivered as follows:

Number of math classes per week	Year			
	1.	2.	3.	4.
before 1995	4	5	5	5
before 2008	4	5	5	5
after 2008	4	4	3	3

(The weekly number of classes after 2008 refers to a minimum required by the state curriculum. It may differ depending on the school curriculum.)

The analysis of the last twenty years of mathematical education in primary school in the areas of arithmetic and algebra shows that these areas of mathematics were affected by some reductions in their content with respect to the first stage of education.

The pupil upon completion of year four of primary school in the period before 1995, had acquired the range of natural numbers to million and over million and he knew, within this range, how to add, subtract and multiply (both from memory and in written algorithm); he knew how to divide with a single-digit divisor using written algorithm; knew Roman numerals; could read and write fractions (notions as fraction bar, numerator and denominator).

After 1995 there was little change regarding the area of arithmetic and algebra. The content remained unchanged, except for fractions. The curriculum from 1995 [10] did not explicitly mentioned fractions; the tasks on propedeutics of fractions, however, appeared in the textbooks from this period. In the section of recommended expanded curriculum some new topics were offered as a possibility to enrich mathematical education, for example some exercises on multiplication with combinatorial motivation or calculations performed on the calculator.

But quite different situation occurred after 2008, when the state program of education for primary stage entered into force. A hypothetical situation may happen that a pupil who completes first fourth years of primary school under the new program would know the numeric range to only 10 000, would multiply and divide only within multiplication table, would not master written algorithms to neither multiply nor divide multi-digit numbers by a single-digit divisor and would not know Roman numerals nor concepts related to fractions. On the other hand, the thematic unit Solving Applied Tasks and Tasks which Develop Specific Mathematical Thinking offers in each grade additional topics for enriching instruction in mathematics.

From the above it is evident that the scope of primary stage arithmetic and algebra has been reduced and that the pupil progresses to the 2nd stage (lower secondary stage) of basic school with less knowledge to be able to build on in the following cycle of mathematical education. Reduction of the curriculum content, however, did not result from its difficulty for the junior school age pupil, but was the result of a reduction in the number of classes prescribed for primary stage mathematics.

Also, the body of knowledge from geometry prescribed for the first fourth years of primary school was reduced between 1995 and 2008 (see [10]). Compared with the previous period the student did not have to address the relative position of two lines in a plane, did not work with the concepts of intersecting lines, the intersection of two intersecting lines, parallel lines, parallelograms; angle (side, vertex, bisector). He might not have met with graphical sum, difference, multiple line segments, area of rectangles (only in the recommended expanded syllabus) or units of area.

Further reduction of the geometric syllabus in the primary stage since 2008 has been brought by the *State Program of Education, Mathematics (Learning Area: Mathematics and Work with Information), Annex ISCED 1* ([9]). The pupil who completes first fourth years of elementary school after 2008 is expected to know only the following notions from geometry: point, line segment, circle (centre, radius), circle, square, rectangle and quadrilateral. He will not necessarily know the terms as perpendicular, right angle and will not draw perpendicular lines or rectangles using a drafting triangle. Constructing formations from blocks according to a pattern, picture or plan and drawing plans for constructions is a new topic introduced by the curriculum. (The above topic, however, appeared in the primary mathematics textbooks in the period between 1995 – 2008 even though it was not explicitly mentioned in the curriculum [10]).

The analysis of the portions of geometry included in mathematical education in primary school in the previous twenty years shows that the given area of mathematics was affected by a significant limitation of its content.

From the given analysis it is evident that the scope of geometry is significantly reduced in the curriculum and that the pupil possesses rather limited range of knowledge to build on in geometry at the outset of the 2nd stage of basic school. Reduction of the geometric content of the curriculum, however, did not result from its difficulty for the junior school age pupil, but analogously to the fields of arithmetic and algebra, it was due to a reduction in number of mathematical classes prescribed for primary stage.

Before 1995, the mathematical education in primary school focused on topics from arithmetic, algebra and geometry. No additional or expanded curriculum was explicitly mentioned in the educational materials. Thus, topics (or individual tasks) from other areas of mathematics did not occur, in most cases, in teaching mathematics in the 1st stage of basic school.

The curriculum of 1995 [10] divided the content of mathematics into basic and advanced categories. The number of classes designated for basic content did not take up the full capacity of classes intended for the given year. Surplus classes could then be used to reinforce the thematic units of the basic curriculum, which was the case in majority cases of the teaching practice. Alternatively, they could be used to cover the topics from the expanded curriculum. In such case, the following topics from the recommended expanded curriculum were included into the teaching process: introduction to the syllabus of logic, equations, number line - representation of numbers, indirectly formulated verbal tasks, unknown in the expression, sorting geometric shapes according to their characteristics, developing functional thinking, the unit of length in the past, notation of the expansion of the natural numbers in decimal system, more difficult inequations, verbal tasks with non-empty intersection, constructing a triangle given the sides, determining the points on the square grid, converting the mixed units, numeric performances on calculator, approximate counting with rounding numbers, sorting by two properties, the area of triangle (square, rectangle) in square grid, drawing a square (rectangle) using drafting triangle.

A single topic from the other areas of mathematics, included in the recommended expanded curriculum, was solving tasks on multiplication with combinatorial motivation. The mathematical textbooks for the first to fourth year of basic school contained various combinatorial problems and elementary tasks from graph theory (connected line graphs, mazes, colouring maps). [7]

Different approach to the content of mathematical education was chosen in 2008 (according to [9]). There is a thematic unit Solving Applied Tasks and Problems Developing Specific Mathematical Thinking for each year of primary stage. It is not included as an extra or optional part of the curriculum (as a recommended expanding curriculum in the previous period). It is an integral part of mathematical curriculum in the first fourth years of basic school and is listed in recommended content and performance standards.

The range of mathematical education in the first stage of basic school has been extended by some new topics (even from the fields other than arithmetic, algebra and geometry) which were absent in the previous period: propedeutics of probability (probabilistic games, experiments and observations and the types of events), collecting and grouping data, creating tables and bar charts, propedeutics of arithmetic mean and combinatorial problems (in greater extent).

This enrichment of mathematical instruction by new topics (as early as in the primary stage) has brought a greater dimension for applied mathematics and was probably a reaction to the content of mathematics that occurs in international surveys of mathematical abilities.

Prospective Teachers for Primary Education

In an environment of tertiary education facilities (teacher training faculties) which prepare teachers for primary education, several surveys were conducted in the last twenty years in which the mathematical abilities of students –trainees were surveyed from different perspectives. Each of the survey's results were analysed with regard to the type of secondary school completed by the trainee.

Academic year 1997/1998, Faculty of Education University of Prešov, breakdown by the number of students who completed the particular type of secondary school (after [6]):

- Grammar School - 52 %
- Secondary Pedagogical School - 23%
- Secondary Trade School - 22%
- Secondary Vocational School - 3 %

Academic year 2006/2007, Faculty of Education University of Prešov, breakdown by the number of students who completed the particular type of secondary school (after [5]):

- Grammar School - 24 %
- Secondary Pedagogical School/Academy of Education and Social Work - 27 %
- Secondary Trade School - 42 %
- Secondary Vocational School - 7 %

Academic year 2008/2009, Faculty of Education University of Prešov + Faculty of Education UMB in Banská Bystrica + Faculty of Education, Trnava University (after [4]):

- Grammar School - 24 %
- Secondary Pedagogical School - 36 %
- Secondary Trade School - 29 %
- Secondary Vocational School - 7 %

Academic year 2013/2014, Faculty of Education, University of Prešov (according to the admitted applicants's data):

- Grammar School - 32%
- Secondary Pedagogical School - 25 %
- Secondary Trade School - 43 %

The above data provide a partial view on how the composition of students - prospective teachers changed over twenty years according to the type of completed secondary school. The types of secondary schools did no fully comply in each of the individual surveys. However, certain trends can be identified:

- drop in interest for primary teacher training among grammar school graduates,
- stable number of secondary pedagogical school graduates (or educational and social academy),
- increase in the number of graduates of secondary trade schools or secondary vocational schools.

Trainees' mathematical knowledge, skills and experience with which they come to the faculty are important factors for their further mathematical preparation. These are partly contingent to the curricula of mathematics at various secondary schools. Gerová [2] indicates the proportion of mathematics lesson:

	Grammar School				Secondary Pedagogical School/ AoEaSW				Business Academy				Secondary Vocational School, 4-year			
	1.	2.	3.	4.	1.	2.	3.	4.	1.	2.	3.	4.	1.	2.	3.	4.
Number of Classes before 2008/9	4	4	3	3	3/2	3/2	2	-	3	3	2	2	2/3	2	2	2/3
Number of Classes after 2008/9	4	4	3	1	2	2	2	-	3/2	3/2	2	-	2	2	1/2	1/2

(Weekly hours may vary depending on the type of secondary schools and its school curriculum.)

The above facts, the decrease in interest among grammar school graduates in teacher training for primary stage and the decline in the number of math classes in secondary schools are factors that must be taken into account when designing an undergraduate training of prospective teachers in the area of mathematics.

Within the grant project VEGA *Analysis of Mathematical Preparation of Students of Preschool and Elementary Education from the Perspective of the Development of Mathematical Literacy* the research was conducted (the sample included three faculties in Slovakia) which showed that the students at the outset of their study had major deficiencies in mathematical abilities. This finding does not provide a good starting base indeed for subject-specific training of prospective primary teachers in mathematics.

Achievements of Slovak Pupils in the TIMSS survey

TIMSS (*Trends in International Mathematics and Science Study*) survey examines in four-year cycles the knowledge of pupils in mathematics and science. Slovak pupils were involved in testing at the end of the year one of basic school between 2007 and 2011 attaining the following results:

2011 51 countries ***2007 36 countries***	**Average Score/Ranking of SR**							
	Overall Results		***Content Domain* Number**		***Content Domain* Geometric Shapes and Measures**		***Content Domain* Data Display**	
	2011	**2007**	**2011**	**2007**	**2011**	**2007**	**2011**	**2007**
Slovak Republic	507/25.	496/21.	511	495	500	499	504	492
OECD Countries Average	521	-	520	-	522	-	525	-
EU Countries Average	519	-	519	-	519	-	521	-
TIMSS Average Scale	500	500	494	500	484	500	484	500

The results of 2007 inform that the Slovak pupils rank statistically between the 16th - 25th place within in the countries (36) involved in TIMSS 2007 survey. Comparing the achievement of Slovak pupils with the average of the European Union (EU) countries and/or OECD (Organisation for Economic Co-operation and Development) countries, position of the Slovak Republic was below the average of these countries.

In the study by TIMSS 2011 – Mathematics, Slovak students achieved the result significantly below the average of the participating EU countries as well as OECD countries. It was the 25th position out of the 51 countries in the ranking. The comparison of the changes in the results over the time - trends in pupils' achievement - shows that from a statistical point of view the result of the 2011 is comparable with that of the 2007 study. ([1], [3])

It would not be appropriate, however, to view the achievement of Slovak pupils and the conclusions resulting from it as a totally negative image of mathematical education in the primary stage in Slovakia. The TIMSS tests contain some tasks that are difficult for the Slovak pupils from the aspect of content because they encounter similar tasks only at the second (lover secondary) stage of basic school.

In 2007 and 2011, there were tested Slovak pupils whose mathematical education at the primary level was delivered after the curriculum of 1995 [10]. Another TIMSS study will be implemented in 2015. If Slovakia participates, it will concern the students who will be educated in the 1st stage of basic school complying with the State program of Education. It will be interesting to explore which changes in primary mathematical education would have a greater impact on the achievement of Slovak pupils. Whether it is of any importance that the reduction of traditional curricular content and the reduction in the number of mathematical classes would be reflected in the results of Slovak pupils by that their achievement, in an international comparison, would be worse than in 2007 and 2011. Or, on the other hand, would the inclusion of new topics into mathematical education in primary school contribute to improved results of Slovak pupils.

Conclusion

What are the external factors that influence mathematical abilities of students at the end of the primary stage of education? Firstly, it is the content of mathematical education which is defined in the curricular documents.

Very important (and perhaps the most important) factor seems to be the teacher. His professional and educational competences significantly determine the educational achievements of pupils in general and certainly in mathematics. In this context, it is thus crucial what the undergraduate mathematical training of prospective teacher is like.

Non negligible factor is a societal perception of mathematical education. If a child in the family, neighbourhood and mass-media encounters negative attitudes towards mathematics it is then very difficult to implement efficiently the process of teaching mathematics in school.

One approach that could make teaching mathematics more attractive is to search for and find some positive examples of practice not only in Slovak context but also from foreign countries. Given the ranking of individual countries in the TIMSS study, it is imperative to analyse mathematical education in those countries where students achieve good results in mathematics and to compare foreign and Slovak curricular documents for primary mathematics so as to get inspiration and interesting suggestions.

All effort should be directed towards optimising mathematical abilities of students. But, perhaps even more important aspect is that the attitudes of pupils towards mathematical education were positive at the end of the primary stage of school. That would be the best investment for further education.

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ON A CERTAIN GROUP OF LINEAR SECOND-ORDER DIFFERENTIAL OPERATORS OF THE HILL-TYPE

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ABSTRACT. *The Hill differential equation often occurs in physical, technical and astronomical topics, in particular in modelling connected with vibrations of mechanical systems. It is a special equation in Jacobi form investigations of which is motivated by concrete modelling time functions. In the contribution there is also established isomorphism between a certain group of Hill type differential operators and the subgroup of the group of third order differential operators with constant coefficients.*

KEY WORDS: *group, binary operation, differential equation, differential operator*

CLASSIFICATION: *H40, I40, 47E05*

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The application of algebraic methods in the theory of linear differential equations offers new views on the qualitative theory of differential equations and provides to reveal new interesting connections which can lead to new valuable results. As it was shown in appropriate literature [3, 11, 12], in the 1950s professor Otakar Borůvka started a systematic study of global properties of linear differential second-order equations. Following Borůvka´s approach, František Neuman obtained further significant results, which generalized previously acquired results. This topic was also dealt with by other members of Borůvka´s differential equations seminar. For the constituting scientific branch there is characteristic the application of not only analytical methods used while examining differential equations, but also the usage of algebraic, topological and geometrical tools. Local methods and results are not sufficient when problems of global nature are studied, e.g. boundedness, periodicity, asymptotic and oscillatory behaviour of equations solutions, factorization of operators formed of left sides, and other properties.

In this article, which follows up the topic of a series of works devoted to algebraic properties of linear differential operators formed of left sides of linear differential equations, including the properties of groups and hypergroups of their solution spaces, we deal with the hypergroup of operators formed of operators – the left sides of second-order linear differential equations of the Hill-type. These are equations within the Jacobi form, specially equations with periodic coefficients. The Hill equation is of the form

$$y'' + \left[\Phi(x) + \lambda\right] \cdot y = 0 \ , \tag{H}$$

with periodic function $\Phi(x)$. The differential equation of this type often occurs in physical, technical and astronomical topics, especially while solving problems connected with vibrations of mechanical systems.

In monography [8], p. 411 there is contained the Hill equation (as example 2.20) in the form

$$y'' + (ae^{2x} + be^{x} + c)y = 0 \tag{1}$$

with the periodic function $\Phi(x) = ae^{2x} + be^{x}$ with the complex period $2\pi\, i$. After the transformation $y(x) = u(t)$, $t = ix$, we obtain the above equation (H). Indeed for $y(x) = u(ix)$, $t = ix$ we have

$$u''(t) - (ae^{-2ti} + be^{-ti} + c)u(t) = 0,$$

thus $$u(t) + [\Phi(t) + \lambda]\cdot u(t) = 0,$$

where $\Phi(t) = -ae^{-2ti} + be^{-ti} = -a\cos 2t - b\cos t + (a\sin 2t + b\sin t)\,i$, which is a periodic function with the basic period 2π and $\lambda = -c$.

Using the substitution $y = u(t)e^{-\frac{1}{2}x}$, $t = e^{x}$ we obtain from (1) the equation

$$t^2u''(t) + (at^2 + bt + c + \frac{1}{4})u(t) = 0, \tag{2}$$

(cf. [8] p. 411), which is a differential second-order equation with polynomial coefficients. Moreover the above presented equations (1) and (2) are linear differential second-order equations in so called Jacobi form. It has been mentioned in [1, 6], Otakar Borůvka has obtained a criterion of a global equivalence for the second-order differential equations within the Jacobi form and he also found corresponding global canonical forms for such equations. For more information see F. Neuman [11, 12]. Notice, that second-order linear differential equations we also obtain as equations of functions modelling certain time processes. For example, consider the functions of the Gaussian-shaped pulse signal $v(t) = a\exp(-2\pi t^2)$, $t \in \langle 0,\infty)$. Corresponding differential equation has the form

$$v''(t) - 16a\pi t^2 v(t) = 0,\ \ t \in \langle 0,\infty). \tag{3}$$

with initial conditions $v(0) = a, v'(0) = 0$.

For the shape analysis of non-periodical time signals (or impulses determined by radiation) there is used in [7] product of simple quadratic or cubic polynomials with exponential functions. There are functions

$$\psi(t) = at^2\exp(-\lambda t), \varphi(t) = at^3\exp(-\lambda t),\ \ t \in \langle 0,\infty).$$

Considering the general modelling time function

$$\varphi(t) = t^n\exp(-\lambda t), n = 2,3,\ldots,\ \ t \in \langle 0,\infty)$$

we obtain the second-order equation in the Jacobi form

$$\varphi''(t) + p(t)\varphi(t) = 0, \tag{4}$$

where $p(t) = (-\lambda^2t^2 + 2\lambda nt + n(n-1))\,t^{n-2}$, $t \in \langle 1,\infty)$

with initial conditions $\varphi(1) = \exp(-\lambda), \varphi'(1) = (n-\lambda)\exp(-\lambda)$. The equation (4) can be rewritten into the form

$$t^{n-2}\varphi''(t) - (\lambda^2t^2 - 2\lambda nt + n(n-1))\,\varphi(t) = 0,$$

which is a second-order linear differential equation with polynomial coefficients.

Finally, the modelling time function $y(t) = A(1-\exp(-ct))^b$ called the Chapman-Richardson`s function, which is one of the most common functions based on the original Bertalanffy equation derived for growth and increment of body weight, leads also to second-order differential equation of the Jacobi form – cf. [9, 10].

Algebraic properties of structures formed by ordinary differential operators of the second order and n-th order as well (formed of left hand sides of corresponding homogeneous differential equations) have been studied in several papers. Let us mention at least [1, 2, 4, 5, 6, 9, 10, 12].

Suppose, $p, q: I \to \mathbf{R}$ are continuous functions,

$$L(p, q)\, y = y'' + p(x)\, y' + q(x)\, y,\ x \in I \subseteq \mathbf{R},\ y \in \mathbf{C}^2(I)$$

is a second order differential operator. Denoting

$$\mathbf{LA}_2(I)_q = \{L(p, q);\ p, q \in \mathbf{C}(I),\ q(x) \neq 0\},$$

$$\mathbf{JA}_2(I)_q = \{L(0, q) \in \mathbf{LA}_2(I)\ ;\ q \in \mathbf{C}(I),\ q(x) \neq 0\},$$

we have according to [1] (or theorem 10, [6]) that for the binary operation

$$\bullet_B : \mathbf{LA}_2(I)_q \times \mathbf{LA}_2(I)_q \to \mathbf{LA}_2(I)_q$$

defined by $$L(p_1, q_1) \bullet_B L(p_2, q_2) = L(p_1 q_2 + p_2,\ q_1 q_2)$$

the groupoid $(\mathbf{LA}_2(I)_q, \bullet_B)$ is a non-commutative group with the unit $L(0,1)$ assigning to any function $f \in \mathbf{C}^2(I)$ the function $f'' + f$. Further, if we denote

$$\mathbf{J}_C\mathbf{A}_2(I)_q = \{L(0, r)\ ;\ r \in \mathbf{R},\ r \neq 0\}$$

then with respect to Theorem 2 [6] we have that the subgroupoid $(\mathbf{J}_C\mathbf{A}_2(I)_q, \bullet_B)$ of the group $(\mathbf{JA}_2(I)_q, \bullet_B)$ is its normal commutative subgroup. Other details can be found in the paper [6].

For any triad of real or complex numbers $a, b, c \in \mathbf{R}\ (\mathbf{C})$ define

$$L(0, (a, b, c))\, y = y'' + (ae^{2x} + be^{x} + c)\, y,\ y \in \mathbf{C}^2(I)$$

and put $\mathbf{H}_2(I) = \{L(0, (a, b, c));\ a, b, c \in \mathbf{R}\ (\mathbf{C}),\ c \neq 0\}$. There is possible to define various binary operations on the set $\mathbf{H}_2(I)$. The following possibility is in connection with linear differential operators of the third order – [2, 5].

Suppose $L(0, (a_0, a_1, a_2)), L(0, (b_0, b_1, b_2)) \in \mathbf{H}_2(I)$, with $a_2 \neq 0 \neq b_2$. Define

$$L(0, (a_0, a_1, a_2)) \circ L(0, (b_0, b_1, b_2)) = L(0, (a_2 b_0 + a_0,\ a_2 b_1 + a_1,\ a_2 b_2)).$$

It can be easily verified that the above operation creates on $\mathbf{H}_2(I)$ a structure of non-commutative group with the unit $L(0, (0,0,1))$. If $L(0, (a, b, c)) \in \mathbf{H}_2(I)$, then the inverse element to this operation is $L^{-1}(0, (a,b,c)) = L\left(0, \left(-\frac{a}{c}, -\frac{b}{c}, \frac{1}{c}\right)\right)$. Indeed

$$L(0,(a,b,c)) \circ L^{-1}(0,(a,b,c)) = L(0,(a,b,c)) \circ L\left(0,\left(-\frac{a}{c},-\frac{b}{c},\frac{1}{c}\right)\right) =$$

$$L\left(0,\left(-\frac{ca}{c}+a,-\frac{cb}{c}+b,\frac{c}{c}\right)\right) = L(0,\ (0,0,1)).$$

The group $(\mathbf{H}_2\,(I),\ \circ)$ is isomorphic to the group $(\mathbf{L}_C\ \mathbf{A}_3(I), \circ_2)$ of linear third-order differential operators with constant coefficients. These operators are of the form

$$L(p_0,\ p_1,\ p_2)\ y(x) = y'''(x) + \sum_{k=0}^{2} p_k y^{(k)}(x)\ .$$

The group $(\mathbf{L}_C\ \mathbf{A}_3(I), \circ_2)$ is a subgroup of the group $(\mathbf{L}\ \mathbf{A}_3(I), \circ_2)$ which is investigated in [2, 5]. The corresponding isomorphism $F : (\mathbf{L}_C\mathbf{A}_3(I), \circ) \to (\mathbf{H}_2\,(I), \circ)$ is defined by

$F(L(p_0,\ p_1,\ p_2)) = L(0,\ (p_0,\ p_1,\ p_2))$ for any triad $[p_0,\ p_1,\ p_2] \in \mathbf{R} \times \mathbf{R} \times (\mathbf{R} - \{0\})$.

In fact to each third-order differential operator

$$L(p_0,\ p_1,\ p_2) = \frac{d^3}{dx^3} + p_0\frac{d^2}{dx^2} + p_1\frac{d}{dx} + p_2 Id$$

there is assigned the operator

$$L(0,\ (p_0,\ p_1,\ p_2)) = \frac{d^2}{dx^2} + (p_0 e^{2x} + p_1 e^{x} + p_2)Id$$

of the Hill-type with $p_k \in \mathbf{R}$, $p_2 \neq 0$.

The simplest binary operation on $\mathbf{H}_2\,(I)$ is motivated by direct products of triads of real numbers which leads-after application of so called Ends-lemma-[13] to binary hyperstructures. Considerations of this direction deserve to be investigated in a separate paper.

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ACTA MATHEMATICA 17

ON PENCILS OF LINEAR AND QUADRATIC PLANE CURVES

MARTIN BILLICH

ABSTRACT. *There is a range of geometry software tools for teaching and learning many topics in geometry, algebra and calculus from middle school to the university level. Dynamic software package like Geogebra help students to acquire knowledge about many geometric objects both in the classroom and at home. We show basic facts from the theory of pencils of planar curves that can be defined as a linear combination of their equations. In addition, we look at some possibilities of the software GeoGebra to visualize linear and quadratic plane curves, and their linear combinations. In this paper, possible approach how to use the pencil of planar curves in solving classical problems of analytic geometry is discussed.*

KEY WORDS: *Pencil of planar curves, linear combination, dynamic software.*

CLASSIFICATION: *G44, D44*

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1 Introduction

GeoGebra is a multi-platform mathematical software that is simple to use and which is successfully applied at all levels of education in geometry, algebra and calculus. Dynamic geometry software can help students to explore and understand more concepts in geometry on their own. They may see with their own eyes that this or that geometrical theorem is true. By computers with dynamic geometry software we are also able to solve problems which are difficult to solve by classical approach.

In school mathematics, often the applications of a well-known statement are more important than the verification the statement itself. Therefore, we can start with a very simple problem that hopefully both demonstrates the interplay between algebra and geometry and gives motivation to study problems in which understanding of "why it works" is necessary.

Problem 1. Find all solutions to the following system of equations:

$$x+2y-4=0\,;\;\; 3x-5y-1=0 \tag{S}$$

Solution. Multiplying left side of the first equation by 5 and left side of the second by 2, then adding the results and dividing by 11 we obtain $x-2=0$ and, analogously, multiplying left side of the first equation by -3 and adding corresponding sides of both equations and dividing by -11, we have $y-1=0$. Therefore (2, 1) is the solution of (S).

How can we be sure that, both following system of equations

$$x+2y-4=0\,;\;\; x-2=0 \tag{S$_1$}$$

$$x+2y-4=0\,;\;\; y-1=0 \tag{S$_2$}$$

are equivalent to (S), i.e. have the same set of solutions as (S)? Using a geometric interpretation of the system (S), the question can be formulated as "why do lines whose equations are $x+2y-4=0$, $x-2=0$ and $y-1=0$ intersect the line $3x-5y-1=0$ at the same point?”

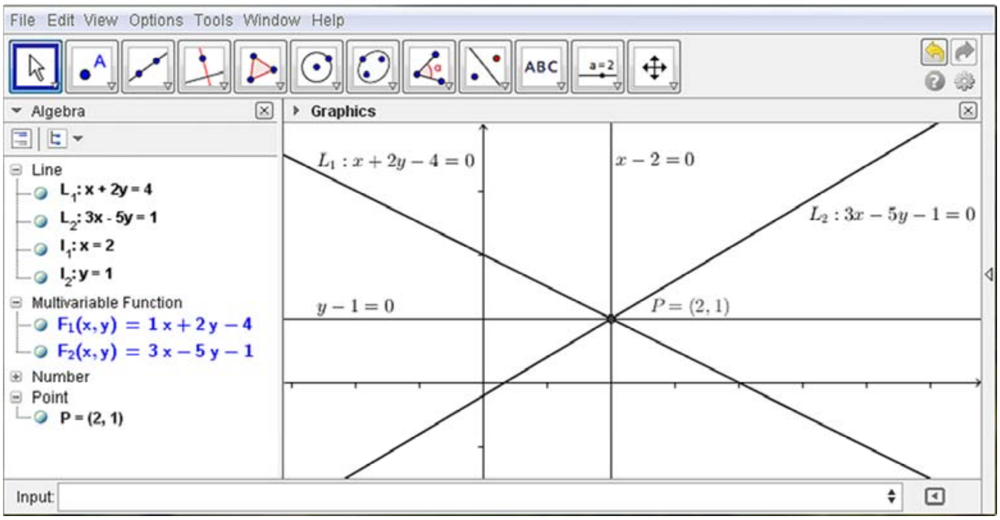

Figure 1

The method we are using above is well-know for everyone. By dynamic software GeoGebra we may verify this method asking a question whether the intersection points of two pairs of lines whose equations are (S_1) and (S_2) are coincident (see Figure 1). Remember that this verification does not replace a proof. We can state the following theorem [1]:

Theorem 1. *Let $F_1(x,y)$ and $F_2(x,y)$ be two algebraic expressions defined on a set S, $S \subset R^2$, and $\lambda, \mu \in R$, $\mu \neq 0$. Then the following systems*

$$F_1(x,y)=0, \qquad F_2(x,y)=0; \tag{I}$$

$$F_1(x,y)=0, \qquad \lambda F_1(x,y)+\mu F_2(x,y)=0 \tag{II}$$

are equivalent.

In terms of the geometrical interpretation of (I) and (II), the theorem says that the curves defined by the equations $F_2(x,y)=0$ and $\lambda F_1(x,y)+\mu F_2(x,y)=0$ intersect the graph of the curve with the equation $F_1(x,y)=0$ at the same set of points.

When L_1 and L_2 are intersecting lines given by equations $L_1(x,y)=0$, $L_2(x,y)=0$, then the linear combination

$$\lambda L_1(x,y)+\mu L_2(x,y)=0, \tag{1}$$

where λ and μ are parameters not both zero, is the equation of the *pencil of lines* that contain the unique intersection point of L_1 and L_2.

2 Pencil of conics

Let us consider two intersecting lines L_1 and L_2 defined by the equations $L_1(x,y)=0$, $L_2(x,y)=0$. Can we describe the family of all conics passing through the intersection of L_1 and L_2 ? The main idea is to again look at linear combinations of L_1 and L_2 , but now, the coefficients will be linear polynomials. That is,

$$\lambda(x,y)L_1(x,y)+\mu(x,y)L_2(x,y)=0 \tag{2}$$

where $\lambda(x,y)$ and $\mu(x,y)$ are polynomials in two variables of degree one, covers all the conics passing through the intersection of L_1 and L_2 .

A GeoGebra environment allows the definition of functions of two variables, so we can draw the graph of the implicit function (2). For the polynomials $\lambda(x,y)$ and $\mu(x,y)$, determined by equations of the form $ax+by+c=0$, we can create constants *a*, *b* and *c* with *"Sliders"*. Then we can change the values of *a*, *b* and *c* for both polynomials and see the changed conic described by (2) in both the geometry and algebra windows. In this way, we can determine the family of conics that contains ellipses (including the special case of circles), parabolas, hyperbolas, as well as some degenerate cases such as single point, line, or two lines.

It is also true that the intersection of two distinct conics does not exceed four points. Let C_1 and C_2 be two conics given by equations $C_1(x,y)=0$ and $C_2(x,y)=0$. We call *pencil of conics determined by* C_1 *and* C_2 to the family of conics determined by the quadratic form

$$\lambda C_1(x,y)+\mu C_2(x,y)=0, \text{ where } \lambda,\mu\in R, (\lambda,\mu)\neq(0,0). \tag{3}$$

All the conics of the pencil (3) passing through the four intersection points of C_1 and C_2 , some of which may coincide or be "imaginary".

The same pencil of conics is gotten by replacing one or both the conics $C_1(x,y)=0$, $C_2(x,y)=0$ by two lines. All these lines are determined by distinct pair of points from the set of intersection of the given conics. If we have exactly four common points of the conics and $L_i(x,y)=0$ be the lines joining pairs of these points, then $L_1(x,y)L_2(x,y)=0$ and $L_3(x,y)L_4(x,y)=0$ are (degenerate) quadrics through the four points. So the equation of the pencil (3) can be written in the form

$$\lambda L_1(x,y)L_2(x,y)+\mu L_3(x,y)L_4(x,y)=0 \tag{4}$$

For any pair $(\lambda,\mu)\neq(0,0)$ of values, we have a conic through the four points determined by the equation pairs $L_i(x,y)=0, L_3(x,y)=0$ and $L_i(x,y)=0, L_4(x,y)=0$, for $i=1,2$.

2 Pencils in problem solving

In this part we will be concerned with the above notion of the pencil in some problems which can be easily solved in analytical geometry using an equation of the pencil of plane curves. First we will solve following problem (see [2]).

Problem 1. Prove that if two given parabolas intersect at four points and their axes are perpendicular, then all common points lie on a circle.

Solution. Let P_1 and P_2 be two parabolas whose axes are perpendicular. We have, in a suitable system of coordinates, the equations of both parabolas P_1 and P_2 in the form:

$$P_1(x,y)=a_1x^2+b_1x+c_1-y=0\,,\qquad P_2(x,y)=a_2y^2+b_2y+c_2-x=0\,.$$

With the help of GeoGebra, the configuration of parabolas of given properties has been created (see Figure 2), in which one can use the *"Sliders"* to change the coefficients of polynomials generated both parabolas.

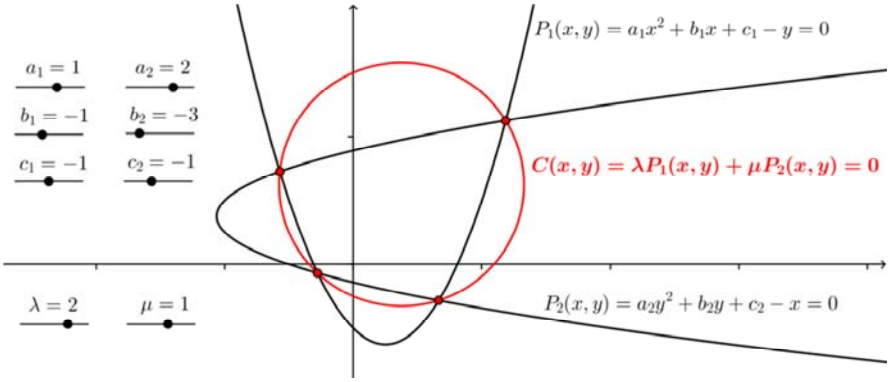

Figure 2

The parabolas P_1 and P_2 give us the pencil

$$\lambda P_1(x,y)+\mu P_2(x,y)=\lambda a_1x^2+\mu a_2y^2+(\lambda b_1-\mu)x+(\mu b_2-\lambda)y+\lambda c_1+\mu c_2=0\,. \qquad (5)$$

Since $a_1\neq 0$ and $a_2\neq 0$, the equation (5) represents the circle passing through the points of intersection of given parabolas if and only if $\lambda a_1=\mu a_2$. We can set $\lambda=a_2$ and $\mu=a_1$ in (5), and the statement is proven.

To show that the points of intersection lie on a circle is very simple with GeoGebra software. It is sufficient to construct a circle that contains three of the four intersection points. In a next step we use a command *"Relation between Two Objects"* for the last of those points and constructed circle. The program says that the point lies on a circle. The same result we can get by using the *"Sliders"* to change the coefficients λ and μ of the conic (5), as we mentioned above, i.e. $\lambda=a_2$, $\mu=a_1$.

Problem 2. Determine the equation of a conic section passing through three given points $A=(2,0)$, $B=(2,1)$ and $C=(-1,2)$ and whose center is $S=(0,0)$.

Solution. As the point S is the center of the symmetry of the conic we are looking for, we can calculate the symmetric points A' of A and B' of B as follows: $A'=2S-A=(-2,0)$, and $B'=2S-B=(-2,-1)$. We have four points of the conic therefore, we can take the lines $y=0$, $x-2y=0$, $x-2=0$ and $x+2=0$ passing through pairs of the points A, A', B, B'. The pencil of conics has equation

$$\lambda(y)(x-2y)+\mu(x-2)(x+2)=0\,.$$

As it contains the point $C=(-1,2)$ we have the following expression:

$$\lambda(2)(-1-2.2)+\mu(-1-2)(-1+2)=0 \quad \Rightarrow \quad 3\mu=-10\lambda$$

If we set $\lambda=3$, than $\mu=-10$, and the equation of the searched conic is

$$10x^2-3xy+6y^2-40=0\,.$$

This curve is an *ellipse* with axes not parallel to the coordinate axes (see Figure 3).

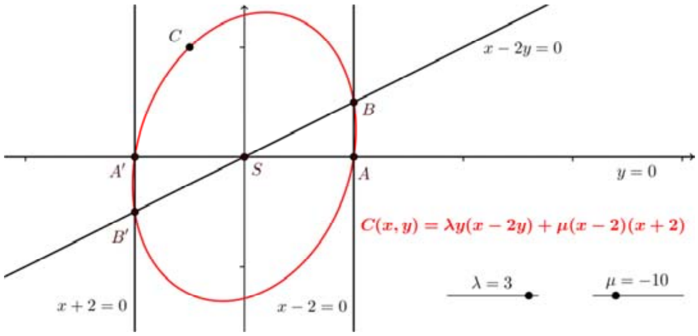

Figure 3

In geometry, we often come across the word *circumcircle* while studying triangles. The circumcircle is the unique circle that passes through each of the triangle's three vertices. We will solve the following problem:

Problem 3. Find the equation of the circumcircle of a triangle whose sides lie on three given lines $L_1(x,y)=x+y-3=0$, $L_2(x,y)=x-y+1=0$ and $L_3(x,y)=x-4=0$.

Solution. Consider a general plane curve given by

$$\lambda L_1(x,y)L_2(x,y)+\mu L_1(x,y)L_3(x,y)+\varepsilon L_2(x,y)L_3(x,y)=0\,, \tag{6}$$

where λ, μ and ε are real numbers. Clearly, any such curve passes through the vertices of the triangle. Therefore, the circle we are looking for has equation

$$\lambda(x+y-3)(x-y+1)+\mu(x+y-3)(x-4)+\varepsilon(x-y+1)(x-4)=0$$

for suitable real coefficients λ, μ and ε. This is equivalent to the following equation

$$(\lambda+\mu+\varepsilon)x^2+(\mu-\varepsilon)xy-\lambda y^2-(2\lambda+7\mu+3\varepsilon)x+(4\lambda-4\mu+4\varepsilon)y-3\lambda+12\mu-4\varepsilon=0\,.$$

Since $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is a circle if and only if $A=C\neq 0$ and $B=0$, and it is not a point or an empty set, λ, μ and ε can be found by equating the coefficients at x^2, y^2 and xy. Hence,

$$\lambda + \mu + \varepsilon = -\lambda\,,\ \ \mu - \varepsilon = 0 \ \ \Rightarrow \ \ \mu = \varepsilon\,,\ \lambda = -\varepsilon\,.$$

If we set $\varepsilon = 1$, than $\mu = 1$, $\lambda = -1$, and the equation of the searched circle is

$$x^2 + y^2 - 8x - 4y + 11 = 0 \ \ \Rightarrow \ \ (x-4)^2 + (y-2)^2 = 9\,.$$

As shown in Figure 4, all the steps above may be demonstrated in the environment of software GeoGebra.

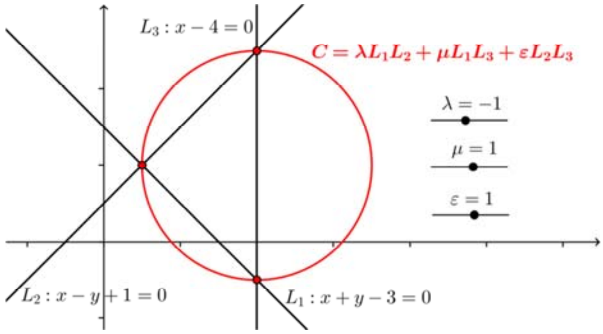

Figure 4

3 Conclusion

The notion of pencil of planar curves that can be defined as a linear combination of their equations is mentioned in this paper. The dynamic software like GeoGebra especially enlightens the connections between algebraic and geometric representation of functions of two variables. In GeoGebra for example it is possible to change the coefficients of linear combination of polynomials determined the pencil of curves. This dynamic point of view allows to students investigate all possible situations for certian family of curves. Although the examples given in this paper are related to the lines and conic sections, the method is also applicable to other plane curves.

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MORE ON FILTER EXHAUSTIVENESS OF LATTICE GROUP-VALUED MEASURES

ANTONIO BOCCUTO, XENOFON DIMITRIOU

ABSTRACT. *We prove some properties of filter exhaustiveness of lattice group-valued measures and give some characterization in terms of continuity of the limit measure. Furthermore we pose some open problems.*

KEY WORDS: *lattice group, (free) filter, (sequential) filter exhaustiveness, (sequential) continuity of a measure.*

CLASSIFICATION: *Primary: 26E50, 28A12, 28A33, 28B10, 28B15, 40A35, 46G10, 54A20, 54A40. Secondary: 22A10, 28A05, 40G15, 46G12, 54H11, 54H12.*

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1 Introduction

The concept of filter exhaustiveness for measures is a powerful and fundamental tool, which has been recently investigated to prove several versions of limit theorems for lattice and topological group-valued measures with respect to filter convergence and some equivalence results (see for instance [3, 4, 5, 7], [8] and the bibliography therein), some comparison results with different kinds of filter convergence (see also [1]), and some results on weak filter compactness and weak filter convergence of measures (see also [2]). In general, it is impossible to obtain results of this kind analogous to the classical ones in the setting of filter convergence, but under the condition of filter exhaustiveness it is possible to give some results about uniform (s)-boundedness, countable additivity or regularity of subsequences, whose elements are indexed by suitable sets of the filter involved.

In this paper we continue the investigation on these topics in the setting of lattice group-valued measures, giving some properties of weak filter exhaustiveness for measure sequences and comparison results, relating them with modes of filter (sequential) continuity investigated in [6] and absolute continuity (see also [10]). In particular, we extend to the context of weak filter exhaustiveness of lattice group-valued measures earlier results of [11], and give some necessary and sufficient conditions for continuity of the limit measure. Moreover we investigate some relation with the continuity of the limit measure of a suitable filter pointwise convergent sequence of measures, and give some equivalence results. Finally, we pose some open problems.

2 Preliminaries

Let G be any abstract nonempty set, R be a Dedekind complete (ℓ)-group, F be a free filter of N, $\Sigma \subset \mathsf{P}(G)$ be a σ-algebra, and $\nu : \Sigma \to [0,+\infty)$, $m_n : \Sigma \to R$, $n \in \mathsf{N}$, be finitely additive measures. Let d_ν be the pseudometric associated with ν, namely $d_\nu(A,B) = |\nu(A) - \nu(B)|$ for each A, $B \in \Sigma$.

Definitions 2.1 (a) An (O)-*sequence* is a decreasing sequence $(\sigma_p)_p$ in R, whose infimum is 0.

(b) A sequence $(x_n)_n$ in R $(O\mathcal{F})$-*converges* to $x \in R$ iff there is an (O)-sequence $(\sigma_p)_p$ in R with $\{n \in \mathbb{N} : |x_n - x| \leq \sigma_p\} \in \mathcal{F}$ for every $p \in \mathbb{N}$.

(c) A sequence $(m_n)_n$ is said to be $(RO\mathcal{F})$-*convergent to* m iff there exists an (O)-sequence $(\sigma_p)_p$ with $\{n \in \mathbb{N} : |m_n(E) - m(E)| \leq \sigma_p\} \in \mathcal{F}$ for every $p \in \mathbb{N}$ and $E \in \Sigma$ (see also [9]).

We now deal with filter exhaustiveness for lattice group-valued measure sequences and continuity for (ℓ)-group-valued measures. Further extensions and developments of these concepts will be given in a forthcoming paper.

Definition 2.2 Let $E \in \Sigma$ be fixed. We say that $(m_n)_n$ is ν-*weakly* $\mathcal{F}$-*exhaustive at* E iff there is an (O)-sequence $(\sigma_p)_p$ such that for any $p \in \mathbb{N}$ there is a positive real number δ such that for any $A \in \Sigma$ with $d_\nu(E, A) \leq \delta$ there is a set $V \in \mathcal{F}$ with $|m_n(E) - m_n(A)| \leq \sigma_p$ whenever $n \in V$.

Definition 2.3 We say that $(m_n)_n$ is *sequentially* ν-*weakly* $\mathcal{F}$-*exhaustive at* E iff there exists an (O)-sequence $(\sigma_p)_p$ such that for each sequence $(E_k)_k$ in Σ with $\lim_k d_\nu(E_k, E) = 0$ and $p \in \mathbb{N}$ there are a sequence $(C_k)_k$ in $\mathcal{F}$ and a set $D \in \mathcal{F}$ with $|m_n(E_k) - m_n(E)| \leq \sigma_p$ whenever $k \in D$ and $n \in C_k$.

Definitions 2.4 (a) A finitely additive measure $m : \Sigma \to R$ is ν-*continuous* at E iff there is an (O)-sequence $(\sigma_p)_p$ in R such that for any $p \in \mathbb{N}$ there is $\delta > 0$ with $|m(A) - m(E)| \leq \sigma_p$ whenever $A \in \Sigma$ and $d_\nu(A, E) \leq \delta$.

(b) We say that m is ν-*absolutely continuous on* Σ iff there is an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ there is a positive real number δ with $|m(A)| \leq \sigma_p$ whenever $\nu(A) \leq \delta$.

Definition 2.5 A finitely additive measure $m : \Sigma \to R$ is said to be *sequentially* ν-$\mathcal{F}$-*continuous at* E iff there is an (O)-sequence $(\sigma_p)_p$ such that for every sequence $(E_k)_k$ in Σ with $\lim_k d_\nu(E_k, E) = 0$ we have $(O\mathcal{F})$ $\lim_k m(E_k) = m(E)$ with respect to $(\sigma_p)_p$.

Remark 2.6 Recall that, given a fixed free filter $\mathcal{F}$ of $\mathbb{N}$ and an element $E \in \Sigma$, a finitely additive measure $m : \Sigma \to R$ is ν-continuous at E if and only if it is sequentially ν-$\mathcal{F}$-continuous at E (see also [6, Theorem 2.1]).

3 The main results

We begin with the following result, which extends [11, Proposition VII.9] to filter exhaustiveness and lattice groups.

Proposition 3.1 *Let $\nu : \Sigma \to [0,+\infty)$ and $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be finitely additive measures, and $E \in \Sigma$. Then the following are equivalent:*

(a) $(m_n)_n$ is ν-weakly $\mathcal{F}$-exhaustive at E;

(b) $(m_n)_n$ is ν-weakly $\mathcal{F}$-exhaustive at $\emptyset$.

Proof: (a) $\Rightarrow$ (b) Let $E \in \Sigma$, $(\sigma_p)_p$ be an (O)-sequence and $\delta > 0$ be according to ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at E. Choose arbitrarily $A \in \Sigma$, with $\nu(A) \le \delta$. Since ν is finitely additive and positive, then ν is monotone too, and hence we get:

$$\nu((E \cup A)\Delta E) = \nu(A \setminus E) \le \nu(A),$$
$$\nu((E \setminus A)\Delta E) = \nu(E \setminus (E \setminus A)) = \nu(E \cap A) \le \nu(A).$$

Let $V \in \mathcal{F}$ be in correspondence with E and ν-weak $\mathcal{F}$-exhaustiveness. Taking into account finite additivity of the m_n's, from (1) for every $n \in V$ we have

$$| m_n(A) | = | m_n(E \cup A) - m_n(E \setminus A) | \le$$
$$| m_n(E \cup A) - m_n(E) | + | m_n(E) - m_n(E \setminus A) | \le 2\sigma_p.$$

(b) $\Rightarrow$ (a) Let $(\tau_p)_p$ be an (O)-sequence and δ be according to ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at $\emptyset$. Pick arbitrarily $E \in \Sigma$, and let $D \in \Sigma$ be with $\nu(E\Delta D) \le \delta$. Then,

$$\nu(E \setminus D) + \nu(D \setminus E) = \nu(E\Delta D) \le \delta,$$

and a fortiori $\nu(E \setminus D) \le \delta$ and $\nu(D \setminus E) \le \delta$. Let V be associated with $E \setminus D$ and $D \setminus E$, thanks to ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at $\emptyset$. For every $n \in V$ we have

$$|m_n(E)-m_n(D)|=|m_n(E\setminus D)+m_n(D\setminus E)|\le$$
$$|m_n(E\setminus D)|+|m_n(D\setminus E)|\le 2\tau_p,$$

getting the assertion.

Analogously as Proposition 3.1, it is possible to prove the following

Proposition 3.2 *Under the same notations and hypotheses as above, the following are equivalent:*

(a) $(m_n)_n$ is sequentially ν-weakly $\mathcal{F}$-exhaustive at E;

(b) $(m_n)_n$ is sequentially ν-weakly $\mathcal{F}$-exhaustive at $\emptyset$.

Remark 3.3 Observe that, arguing analogously as in Proposition 3.1, it is possible to see that a finitely additive measure $m:\Sigma\to R$ is ν-continuous at some set $E\in\Sigma$ if and only if m is ν-continuous at $\emptyset$ if and only if m is ν-absolutely continuous on Σ, and that in this case it is possible to find a single (O)-sequence $(w_p)_p$, independent of $E\in\Sigma$, with respect to which m is ν-continuous at every $E\in\Sigma$.

We now turn to the following characterization of filter exhaustiveness in terms of continuity of the limit measure.

Theorem 3.4 *Let* $E\in\Sigma$, $\nu:\Sigma\to[0,+\infty)$, $m_n:\Sigma\to R$, $n\in\mathbb{N}$, *be finitely additive measures,* $(RO\mathcal{F})$*-convergent to* $m:\Sigma\to R$ *with respect to an* (O)*-sequence* $(\sigma^*_p)_p$, *and fix* $E\in\Sigma$.

Then the following are equivalent:

(a) $(m_n)_n$ is sequentially ν-weakly $\mathcal{F}$-exhaustive at E;

(b) $(m_n)_n$ is ν-weakly $\mathcal{F}$-exhaustive at E.

(c) m is ν-continuous at E.

Proof: (a) $\Rightarrow$ (c) Let $(\sigma_p)_p$ be an (O)-sequence associated with sequential ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at E and $(RO\mathcal{F})$-convergence of $(m_n)_n$ to m respectively, and $(E_k)_k$ be a sequence in Σ, with $\lim_k d_\nu(E_k,E)=0$. In order to prove ν-continuity of m at E, thanks to Remark 2.6 it is enough to show that the sequence $(m(E_k))_k$ $(O\mathcal{F})$-converges to $m(E)$ with respect to the (O)-sequence $(2\sigma^*_p+\sigma_p)_p$.

Choose arbitrarily $p \in \mathbb{N}$. Let $D \in \mathcal{F}$ and $(C_k)_k$ be in $\mathcal{F}$ be according to sequential ν-weak $\mathcal{F}$-exhaustiveness, and pick arbitrarily $k \in D$. By $(RO\mathcal{F})$-convergence of $(m_n)_n$ there is a sequence $(B_k)_k$ of elements of $\mathcal{F}$ with

$$| m_n(E) - m(E) | \vee | m_n(E_k) - m(E_k) | \le \sigma_p^* \quad \text{for any } n \in B_k.$$

For each $n \in B_k \cap C_k$ we have

$$| m(E_k) - m(E) | \le | m_n(E_k) - m(E_k) | + | m_n(E_k) - m_n(E) | +$$
$$| m_n(E) - m(E) | \le 2\sigma_p^* + \sigma_p,$$

getting the assertion.

(c) $\Rightarrow$ (a) By Remark 2.6, m is sequentially ν-$\mathcal{F}$-continuous at E, and hence there exists an (O)-sequence $(\tau_p)_p$ in R such that for every $p \in \mathbb{N}$ and for each sequence $(E_k)_k$ in Σ with $\lim_k \; d_\nu(E_k, E) = 0$ there is $D \in \mathcal{F}$ with $| m(E_k) - m(E) | \le \tau_p$ whenever $k \in D$. By $(RO\mathcal{F})$-convergence of $(m_n)_n$ to m with respect to the (O)-sequence $(\sigma_p^*)_p$, in correspondence with p, E_k, $k \in \mathbb{N}$, and E there exists a sequence $(F_k^*)_k$ in $\mathcal{F}$ with

$$| m_n(E_k) - m(E_k) | \vee | m_n(E) - m(E) | \le \sigma_p^*$$

for any $n \in F_k^*$. Thus for every $k \in D$ and $n \in C_k$ we get

$$| m_n(E_k) - m_n(E) | \le | m_n(E) - m(E) | + | m(E_k) - m(E) | +$$
$$| m_n(E_k) - m(E_k) | \le 2\sigma_p^* + \sigma_p,$$

namely sequential ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at E.

(b) $\Rightarrow$ (c) Let $(\sigma_p)_p$ be an (O)-sequence associated with ν-weak $\mathcal{F}$-exhaustiveness at E, and pick arbitrarily $p \in \mathbb{N}$. By hypothesis there exists a positive real number δ, fulfilling the condition of ν-weak $\mathcal{F}$-exhaustiveness. Fix arbitrarily $A \in \Sigma$ with $d_\nu(A, E) \le \delta$: there exists a set $F_1 \in \mathcal{F}$ with $| m_n(A) - m_n(E) | \le \sigma_p$ for each $n \in F_1$. Moreover, there exists $F_2 \in \mathcal{F}$ with

$$| m_n(A) - m(A) | \vee | m_n(E) - m(E) | \le \sigma_p^*$$

for any $n \in F_2$. For each $n \in F_1 \cap F_2$ we get

$$| m(A) - m(E) | \le | m_n(A) - m(A) | + | m_n(A) - m_n(E) | + | m_n(E) - m(E) | \le 2\sigma_p^* + \sigma_p .$$

Hence $| m(A) - m(E) | \le 2\sigma_p^* + \sigma_p$ for each $A \in \Sigma$ with $d_\nu(A, E) \le \delta$, getting ν-continuity of m at E.

(c) $\Rightarrow$ (b) By ν-continuity of m at E there exists an (O)-sequence (τ_p) in R such that for every $p \in \mathbb{N}$ there is $\delta > 0$ with $| m(A) - m(E) | \le \tau_p$ for every $A \in \Sigma$ with $d_\nu(A, E) \le \delta$. By $(RO\mathcal{F})$-convergence of $(m_n)_n$ to m with respect to the (O)-sequence $(\sigma_p^*)_p$, in correspondence with p, A and E there exists a set $F^* \in \mathcal{F}$ with $| m_n(A) - m(A)$

$$| m_n(A) - m_n(E) | \le | m_n(A) - m(A) | + | m(A) - m(E) | + | m_n(E) - m(E) | \le 2\sigma_p^* + \tau_p ,$$

namely ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at E.

Open problems:
(a) Investigate similar notions of filter exhaustiveness and filter continuity, and prove some related comparison results.
(b) Find different results for measures or functions taking values in different types of abstract structures (for example, metric semigroups or topological groups).
(c) Find similar results considering weaker kinds of convergence.
(d) Study similar results requiring R to be super Dedekind complete or even an arbitrary (ℓ)-group.

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HOW TO CREATE TASKS FROM MATHEMATICAL LITERACY

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ABSTRACT. *In our article we try to present the tasks of mathematical literacy for teachers and students, because we believe that everyone of us should be mathematically literate. We are focused on developing students' competencies to apply basic mathematical thinking and basic skills explore in science and technology. We use The National Educational Program ISCED 3a Mathematics. Tasks are ready for students in grade 13, for high school graduates. These students should already be able to use mathematical knowledge in many different situations in various ways. The tasks are inspired by microbiological knowledge, which can be found in many commercial magazines or in the website.*

KEY WORDS: *mathematical literacy, science literacy, microbiological tasks*

CLASSIFICATION: *D34, D84, M24*

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Introduction

The growing role of science, mathematics and technology in modern life requires that all adults, not just those who aspire to scientific careers, become mathematically, science and technology literate [5]. Research points to the urgency of connecting school mathematics to the outside world [4]. One thing is to know how to calculate the math, the second thing is to understand it, or apply it correctly and interpret current real life situations, encountered, for example, while reading a magazine about health or agriculture.

Students learn to calculate mathematics in specific mathematical situations tasks purely of mathematical content, which lacks real context. For example: students will learn what exponential function are, how to calculate and solve a few examples to practice the mathematical content. The result is that the student controls theory and can computes tasks about the exponential function. However, let us to specify the role of the student with a realistic context and we will be surprised how he will deal with the particular role, and we would be happy if he comes up with a solution. For example, a well-known role of water lilies reads as follows: On the lake, every day lily doubles its surface. 48 days to be grown over the entire surface of the lake water lilies. Which day water lilies grown over half of the lake? 9 out of 10 students will answer that for 24 days. Why? Because the math was not seen in real concept ever before. Motivation decreases, because even though they know math content they are not able to use their knowledge in real life situation. They are losing an important usage of mathematics. The error is not in students, but in the lack of practise use and see mathematics in the real problems, for example in the problem with water lilies on the lake. This issue is completely scrutinized in the study/ The Program for International Student Assessment (PISA). It looks at the needs of the individual manage to solve mathematical problems in which mathematics to practical problem represents a real benefit in the search process solutions [3]. Examines how students are mathematically literate. PISA describes mathematical literacy as: "an individual´s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgments

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and to use and engage with mathematics in ways that meet the needs of that individual´s current and future life as a constructive, concerned and reflective citizen” [3,8].

Created tasks

In this article we present a preview of the three tasks of mathematical literacy. We use The National Educational Program ISCED 3a Mathematics. Tasks are ready for students in grade 13, for high school graduates, because the time allocated for high school graduates is always bigger and thus are expected to manage the solution in tasks. We deal with cross-curricular microbiology. We can say that we also develop scientific literacy. We start every problem with specific task inspired by literature (textbook, scientific journals). We create new task developing mathematical literacy of students. The tasks are inspired by microbiological knowledge, which can be found in many commercial magazines or in the websites [6,11,14]. It is non-standard task for our students. We are inspired by the PISA.

Problem1

<u>Task from the textbook [10]:</u>

The number of bacteria in a culture is counted as 400 at the start of experiment. If the number of bacteria doubles every 3 hours, the number of individuals can be represented by formula $N(t) = 400(2)^{\frac{t}{3}}$. Find the number of bacteria present in the culture after 24 hours.

<u>Our created task:</u>

The number of bacteria *Escherichia coli* in the sample of faecal contamination of water is counted as 1000 at the start of research. The number of bacteria was doubled every 6 hours, because the sample of water was continuously polluted by waste from sewage. Dependence of the number of bacteria over time is expressed in chart shown in Figure 1.

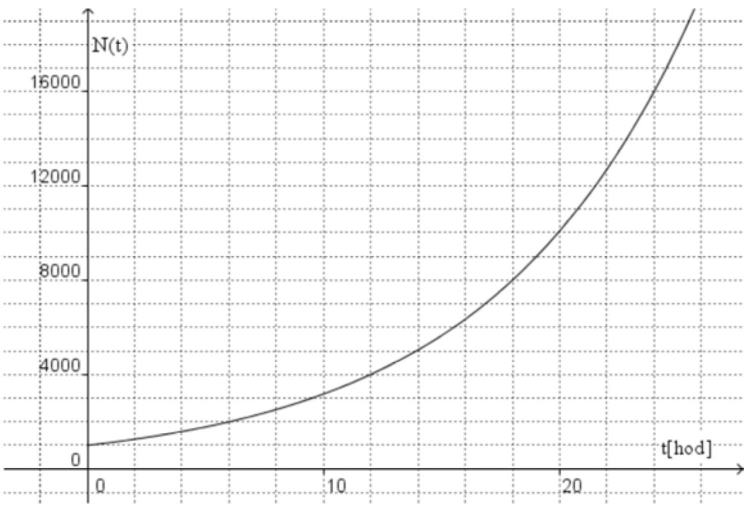

Figure 1: Graph expressing dependence the number of bacteria and the time of growth

Solve the following tasks using the graph shown in Figure 1:

a) Find the number of bacteria $N(t)$ present in culture after 12 hours.

b) Find the time t meanwhile the number of lactic acid bacteria 8-times increased: $N(t) = 8 \cdot N_0$.

c) If a population consisting initially of N_0 individuals also is modeled as growing without limit, the population $N(t)$ at any later time t is given by formula: $N(t) = N_0 \cdot a^{kt}$ (k and a are the constants to be determined). Find the formula for function which graph is shown in Figure 1.

Problem2

We were inspired by the results of the research reported in the scientific journal *Potravinarstvo*. Scientists in the article [11] describe how examining effects of water activity values and incubation temperature on the *Staphylococcus aureus* growth dynamic.

Our created task:

Scientists investigated the effect water activity a_w on the growth dynamic of the bacteria *Staphylococcus aureus*. The values of water activity (the amount of water available for microbial growth) of the tested media were adjusted by *sodium chloride* ($\%NaCl$). The table 1 shows the duration (in hours) of the *lag* phase of growth (where bacteria adapt themselves to growth conditions) and growth rate G_R in the exponential phase (period of growth characterized by cell doubling).

$\%NaCl$	a_w	*lag* [h]	G_R [log KTJ.ml^{-1}.h^{-1}]
0,0	0,999	2	0,320
1,5	0,988	3	0,270
5,0	0,966	6	0,235
8,0	0,955	12	0,230
10,5	0,922	18	0,135
13,0	0,899	36	0,065
15,5	0,877	80	0,040
18,0	0,866	200	0,010

Table 1: Values obtained during experiment (the concentration sodium chloride, the water activity, the duration of lag phase, the growth rate)

Based on the text and the data in table select the incorrect statement:

a) The ability of bacteria to grow at high salt concentrations related to their ability to adapt to osmotic stress during *lag* phase of growth. When the concentration of *sodium chloride* was increased to 5%, there was an extension of the *lag* phase to 6 hours.

b) Water activity a_w is a measurement of the availability of water for biological reactions. It determines the ability of micro-organisms to grow. If water activity a_w decreases, the duration of the *lag* phase to grow will also decrease.

c) Increasing the value of water activity a_w increases the growth rate G_R of the bacteria *Staphylococcus aureus*. Maximum values of the growth rate achieves when water activity is in the range $a_w = 0{,}988$ to $a_w = 0{,}999$.

d) From the data in the table, we can monitor influence of the addition of salt ($\%NaCl$) to the inhibition of bacterial growth. When the *sodium chloride* concentration was increased to 13%, there was the growth rate G_R 5-multiple reduced compared to the growth rate at the beginning of the experiment (0% added salt).

Problem3

We were inspired by the results of the research reported in the scientific journal *Potravinarstvo*. Scientists in the article [6] comparison of occurence lactic acid bacteria in chosen yogurts.

Our created task:

Scientists detected numbers, colony-forming units per milliliter ($CFU.ml^{-1}$), of lactic acid bacteria in yogurts in Slovak markets (creamy yogurts, yogurts with probiotics, non-fat yogurts). In examined yogurts was detected number of lactic acid bacteria ($CFU.ml^{-1}.10^7$) before expiry of the time consumption and after expiry period (see Figure 2). Ratio of the number of bacteria before and after the expiry period was the same for all three types of yogurt.

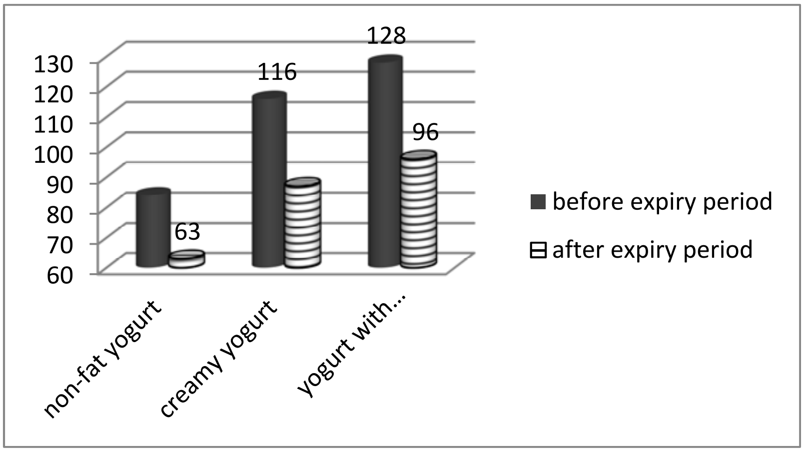

Figure 2: Number of lactic acid bacteria in three types of yogurt, expressed in $CFU.ml^{-1}.10^7$

From the text and data in the Figure 2 solve the following tasks:

a) Calculate the number of lactic acid bacteria in $CFU.ml^{-1}.10^7$ after expiry period in creamy yogurt. Use the data shown in Figure 2.

b) Compare the number of lactic acid bacteria ($CFU.ml^{-1}.10^7$) detected before the expiry period between non-fat yogurt and yogurt with probiotics. How much percent lower was the number of lactic acid bacteria in non-fat yogurt as yogurt with probiotics?

c) Research results were presented on television. The reporter said: "The greatest decrease in the number of lactic acid bacteria was observed in yogurt with probiotics." Do you consider the reporter's statement satisfactory explanation of the graph? Please give reasons for your answer.

Discussion

The mathematics-education community stresses the importance of real-world connections in teaching [4]. One of the main aims of mathematical education as such is preparing the students for dealing effectively with the real-life situations [12]. Real-world connections are expected to have many benefits, such as enhancing students understanding of mathematical concepts, motivating mathematics learning, and helping students apply mathematics to real problems [4].

In modern educational theories is demand for development of not only pupils' knowledge but all key competencies [13]. Mathematical competence involves, to different degrees, the ability and willingness to use mathematical models of thought and presentations [1]. Mathematical competency is defined as insightful readiness to respond to certain mathematical challenges [2]. Competencies in thinking mathematically means "mastering mathematical modes of thoughts" [7]. A competency in posing and solving mathematical problems is about identifying, posing and specifying such problems and solving different problems [9]. When we will develop mathematical competence, then we will also develop the mathematical literacy students. In creating tasks we are focused on developing competencies to apply basic mathematical thinking and basic skills explore in science and technology [1], so apply mathematical thinking to solve practical problems in everyday, uses mathematical models of logical and spatial thinking and presentation and use basic scientific literacy. Students must to work with various representations of relations within solution of presented tasks. The National Educational Program, ISCED 3a, define what the student have to know [1]:

- use different ways of representing mathematical content: text, table, graph (Problem1, Problem2, Problem3),
- read from the graph of the function with sufficient accuracy the size of the functional value and vice versa noted known size of the functional values on a graph (Problem1),
- differentiate exponential dependence and graph exponential functions used for solving of tasks (Problem1),
- write the simple relationships using variables, constants (Problem1),
- determine the unknown value if the specified table relationships (Problem2),
- based on graphic representation to determine an approximate solution – to estimate a solution (Problem3).

Conclusion

The aim of the paper was to highlight the use of mathematics in real situations. We will be glad if these tasks fall into the teaching process, or at least become an inspiration for mathematics teachers in their lessons. Inclusion of presented tasks in the learning process can improve the quality of education and allow to development of mathematical literacy and interdisciplinary students thinking.

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SELECTED GEOMETRICAL CONSTRUCTIONS

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ABSTRACT. *This article deals with selected classical geometric constructions that should be interesting also at the time of modern information and communication technologies. Golden ratio, harmonic proportions, arithmetic and geometrical proportions were already known ratios in the past. Their constructions have been well-known for several centuries. These and many other constructions should not be removed from school curriculums because of their close connection to everyday life. The end of the article is dedicated to a construction, which was in the often used as a layout for building houses last millennium. It is a pity that nowadays these easy constructions have nearly disappeared from common practice. We believe that just these constructions could at this time show closer interconnection between geometry and algebra to students, and thus outline the correlation between synthetic and analytical geometry.*

KEY WORDS: *geometric construction, golden ratio, Pythagorean theorem, harmonic proportion, arithmetic and geometric proportions*

CLASSIFICATION: *E70, F80, G90*

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Introduction

In this age of ICT technologies and the option of graphic software, such as Cabri geometry or GeoGebra, we must not forget that the classical approach to teaching mathematics and geometry should not be completely suppressed by these new technologies.

In doing so, we believe that just these classical constructions could show the students closer interconnection between geometry and algebra and thus could show the students the correlation between synthetic and analytical approach of teaching geometry. Both of these approaches are very important for teaching geometry. Let us demonstrate this on the following theorem:

"Every two conics of the quadratic surface always lie in two conical surfaces."

This theorem was proved using the synthetic method for complexification of a projective space over the field of real numbers. In article [5], we used analytical methods to show that every two conics on the quadratic surface do not always lie on two conical surfaces. The analytical approach allowed us to improve the synthetic one. We discovered a case, when two mutually tangent regular conics, lie on one "simple" conical surface that falls within a sheaf of the quadratic surfaces designated by such conics.

It is apparent that just analytical approach allows us to examine the properties of objects, which we never find by the synthetic approach. We think it is necessary to constantly develop both of these approaches during the whole educational process. Although in the lower grades synthetic approach prevails, the aim should be that students in higher grades know how to solve a complex geometrical problem, utilizing the entire system, i.e. these approaches are not separated, but closely interconnected. Probably every geometry and mathematics teacher wishes that students cope with a geometric situation, so that they are able to plot it and equally well describe it analytically, plus that they know how to determine which approach offers the easiest solution. In our opinion, the following construction could be helpful in doing so.

Golden ratio

Even in the last millennium, people considered basic geometrical constructions necessary and useful, as an evidenced serves the statement by a German mathematician, physicist and astronomer Johannes Kepler (1571-1630):

"There are two treasures of geometry: Pythagoras' theorem and golden ratio. The first has the prize of gold, the other resembles a precious stone."

We have to realize that the ancient knowledge was the basic foundation of modern knowledge and science. Without wisdom and knowledge of classical geometry, there would be no advances of our time, in the form of graphic software. It is true that these provide us completely different options in the teaching of geometry, for example, give us the opportunity to interactively show and demonstrate certain terms. It is, in our opinion, a great asset, but we would consider it a shame if classical methods of teaching geometry completely disappeared.

From this perspective, some geometrical constructions were forgotten over time, especially concerning mathematical studies at secondary schools and universities. Students rarely learn about the golden ratio of a line segment and other important proportions, or about line segments of irrational and rational length. Students should be encountered with plotting these line segments even when plotting on a number axis in a plane or in space ([4]).

The first clear definition of number φ, which was later called the golden ratio, was brought by Euclid around 300 B.C. He formulated a task, which in today's mathematical language could be read as follows: *Divide the line segment AB into two segments - longer AC and shorter CB, so that the area of a square with a side AC is equal the area of a rectangle with sides AB and CB.*

This task can be transformed into another task: Divide the line segment *AB*, so that the ratio of the length the longer part to the shorter is equal the ratio of the entire length of the line segment to the longer part.

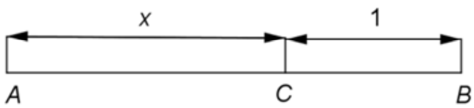

Figure 1: Line segment *AB* divide by golden ratio

If we use the label as in Fig. 1 the task can be written in the mathematically language of algebra in the form of $|AC|:|CB|=|AB|:|AC|$. Without any loss of generality, we can assume that the length of line segment *CB* is $|CB|=1$ and the length of line segment *AC* is x units, $|AC|= x$. Thus can be concluded that $x:1= (x+1): x$. If we find the length of x, the task will be solved. After elementary adjustments, we get that sought length x is the solution of the quadratic equation $x^2 - x - 1=0$.

Solutions of this equation are numbers $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$.

We sought only positive solution to this equation. The searched length of x is given explicitly, i.e. $x = x_1 = \varphi$, which is the value of the golden ratio and we may conclude that the point *C* divides the line segment *AB* by golden ratio [2]. Geometrically, we can construct the point C as shown in Fig. 2. However, this is not the only possible construction of the golden ratio.

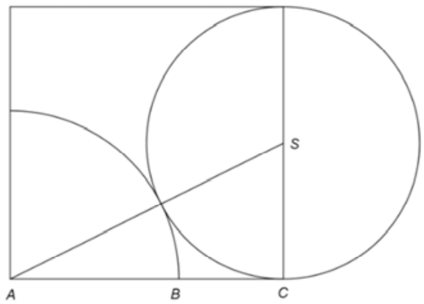

Figure 2: Construction of golden ratio

Harmonic, arithmetic and geometric proportions

Other important ratios that can be considered are arithmetic, geometric and harmonic proportions.

It is a division of a line segment into two parts, so that the length b of the longer part of this line segment is the arithmetic, geometric, or harmonic proportion to the length a of the entire line segment and c the remaining part of this line segment.

The ratios mentioned are now referred to as Pythagorean ratios. If we denote three members a, b and c, where $a > b > c$, then the member b is obtained as arithmetic $b = \frac{a+c}{2}$, $b = \sqrt{ac}$ geometric or harmonic $b = \frac{2ac}{a+c}$ average of the members a and c, i.e. members a, b, c are three consecutive members of an arithmetic, geometric or harmonic sequence.

If we divide a line segment using the arithmetic proportion, the ratio between the longer part and the shorter part is 2:1, the geometric proportion divides the line segment in the Golden Ratio which we have already mentioned, i.e. the ratio between the longer and the shorter part of this segment is $\frac{\sqrt{5}+1}{2}: 1$ $(\varphi: 1)$. By using harmonic proportion the ratio between the length of the longer and the shorter part is $\sqrt{2}: 1$. Construction of the dividing point of the line segment through of all the ratios is shown in Fig. 3, point H denotes the dividing point of line segment in harmonic proportion, point G in a geometric proportion (Golden Ratio) and point A in arithmetic proportion (see article [3]).

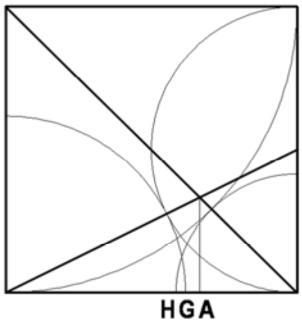

Figure 3: Division of a line segment in arithmetic, geometric and harmonic proportions (from [3])

In the past, these ratios determined the directions of not only music, but also architecture and it is interesting that nowadays students are not able to do these basically simple constructions. No wonder, because nowadays students do not know how to do the constructions of √2, √3 by using so-called Pythagorean triangles or by using Euclid's theorems on the height and the leg of the triangle and what is alarming, they do not know to devise a rational part of the line segment length.

Additional interesting constructions

In this section, we would like to show quite a few other simple constructions with a length of a line segments resulting from the use of a finite number of algebraic operations over the field of real numbers. These constructions can be constructed using compasses and a linear. René Descartes (1596-1650) dealt with this issue as well in his book Geometry. (see [1])

***Example 1*:** Let a and b be the lengths of a line segment. Construct a line segment of the length ab, $\sqrt{a}$, b/a.

Using line segments of the lengths a, b, the line segment of length 1 and the similarity of triangles, we can easily construct the line segment of the length ab, b/a. If we use, for example, Euclidean theorem on the height, we can construct a line segment of length $\sqrt{a}$ as shown in the following figure.

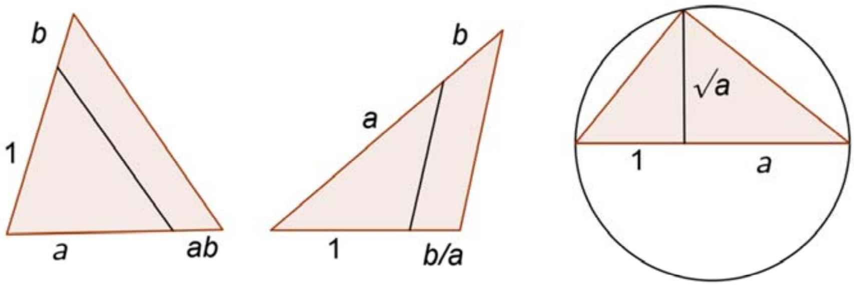

Figure 4: Constructions the line segment of lenghts ab, b/a, $\sqrt{a}$**.**

Furthermore, we would like to mention one more interesting theorem from the book [1], which says that the length of the chord of a circle with a radius of length 1 corresponding with the angle α can be determined by the formula: $d = |AB| = \sqrt{2 - 2\cos\alpha}$.

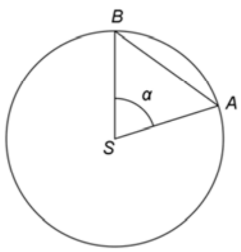

Figure 5

The proof of this theorem is simple and is based on the use of the cosine theorem within an isosceles triangle ASB. This theorem shows that the length of the side of a regular hexagon is equal to the size of the radius of the circle inscribed into this hexagon.

Students learn this particular piece of knowledge in the context of teaching geometry. However, it is unfortunate that they do not learn the general formulation of this theorem, which inter alia implies, that the length of the side of a regular pentagon inscribed in a circle of radius 1 is $\frac{1}{2}\sqrt{10-2\sqrt{5}}$.

***Example 2*:** Let a and b be the lengths of sides of a rectangle. Construct such a line segment whose length is equal to the k-multiple of lengths a, b, where k = 1/2, 1/3, 1/4, 1/5.

Only some of our students know how to structurally solve this task by using the similarity of triangles as shown in Fig. 6 for 1/4.

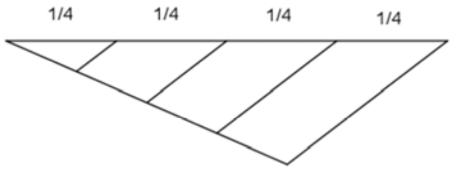

Figure 6: Division of line segment into quarters

It is a great pity that the construction shown in Fig. 7, which was previously massively used in the construction of buildings and the roof trusses, has completely disappeared.

This simple procedure can be proven using the similarity of triangles, which we constructed using this procedure.

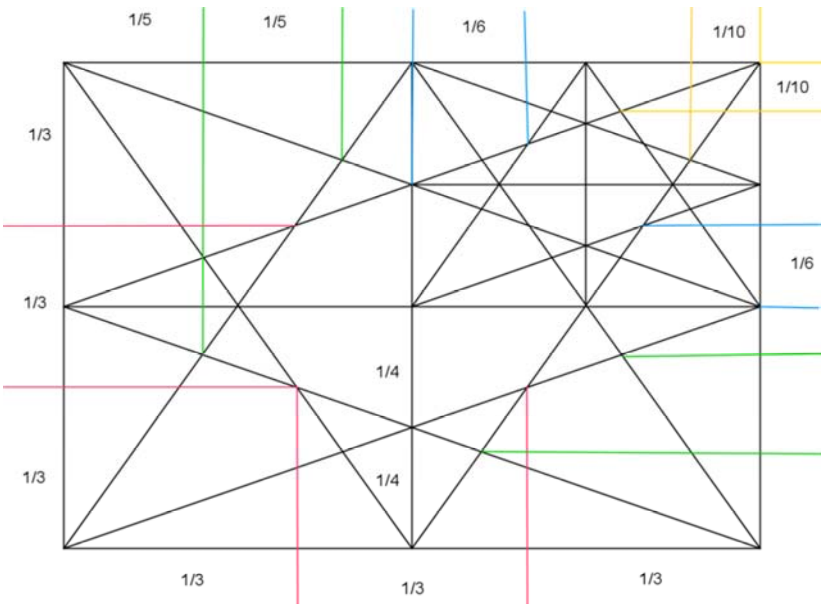

Figure 7: Division of line segment into halves, quarters, thirds, fifths etc.

As displayed in Fig. 7, a further us of this procedure results in more multiples, i.e. sixth, eighth, tenth, twelfth, etc. This construction could be included under the theory of fractals, which has been lately given much more attention.

Conclusion

We can only hope that these days, in age of modern technology, the classical approach and its beauty, which gave us great mathematicians and scholars of our past does not completely disappear from today's teaching of geometry. It would be a shame, if the presented constructions so often used in the past and in practice were completely forgotten.

We think it can help do just that, when preparing future teachers we will be given increased attention of hereby and other interesting constructions, for example, during teaching Euclidean geometry in plane.

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ACTA MATHEMATICA 17

SUDOKU GAME SOLUTION BASED ON GRAPH THEORY AND SUITABLE FOR SCHOOL-MATHEMATICS

VLASTIMIL CHYTRÝ

ABSTRACT. *This article focuses on the logical-mathematical didactic game Sudoku. Analysis of individual fields filling possibilities is mainly based on Graph theory. Ideas, procedures and methods presented in this paper are not demanding and they can be transmitted to secondary school students. In this article the rules of the game and winning strategies analysis derived from Graph theory are mentioned as well as the reasons why this game can be considered a logical game.*

KEY WORDS: *didactic game, Graph theory, winning strategy, logical thinking*

CLASSIFICATION: *D40, A20, E30*

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Introduction

This article is based on the idea that the most natural activity for a student is a game[2] and, therefore, it would be useful for students to use mathematical or didactic games while teaching mathematics[3]. It is a generally known fact that games are often included in teaching for motivational reasons (Vidermanová & Uhrinová, 2011). In this article there are a number of strategies described which can be used when filling the individual fields in Sudoku game. These strategies are primarily based on Graph theory and are so simple that it is possible to use them in teaching this discipline.

Mathematical-logical didactic game Sudoku

The principle of this game is, to some extent, similar to the simple Magic squares (sum of the values in line, column and diagonal is same), which arose as early as about 1000 BC. It is also similar to the Latin Squares (in a square $n \times n$ used n symbols so that in each row and each column each symbol occurs exactly once) from Swiss mathematician Leonard Euler (1707 – 1783). The aim of the game is to fill in the missing numbers 1 – 9 in a predefined, partially pre-filled table which is divided into 9 × 9, i.e. 81 squares. These squares are divided into nine 3 × 3 blocks as figure 1 demonstrates. Filling the table with the numbers must follow these rules:

- Numbers in rows are not repeated
- Numbers in columns are not repeated
- Numbers in 3 × 3 blocks are not repeated

[2] A game is a free action or employment within a clearly defined time and place which is held in freely accepted but, at the same time, unconditionally binding rules. The goal is in itself and it carries with it a sense of tension and joy and, at the same time, awareness of difference from everyday life (Huizinga, 2000).

[3] Martin Gardner defined a mathematical game in Scientific American in a following way: A mathematical game is a multiplayer game whose rules, strategies, and outcomes can be studied and explained by mathematics (Gardner, 1979).

- Order of the numbers when filling is not important

For this game there is no general unambiguous winning strategy. Therefore, the suggestion based on Graph theory how to fill in all the squares will be described. The game should have an unambiguous solution, however, Sudoku without a clear solution are published as well. An example of such a task can be found in fig. 1 where the color-coded numbers are the task. For this task there are four possible solutions that can be obtained by combining possibilities for *A, B, C, D* so that the following is fulfilled:

$$A, B \in \{7;8\} \wedge A \neq B \wedge C, D \in \{5;6\} \wedge C \neq D.$$

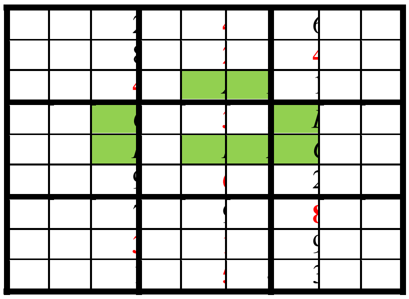

Figure 1: Sudoku without a clear solution

This game can be, to some extent, considered a logical game since while playing it the pupils mainly use the following three logical connections – conjunction, disjunction and consequence. The comment bellow corresponds to 4 × 4 Sudoku in figure 2. Possible considerations could be described in language of logic as follows:

Square *A*: According to the rules this square could be filled with number 3 **or** number 4.

Square *B*: *If* the block rule is ***used together*** with the column rule, ***then*** there is exactly one symbol that can be written in this square.

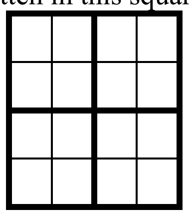

Figure 2: Example of 4 x 4 Sudoku

Sudoku and Graph theory

One of the possible ways to find a winning strategy for Sudoku game is the use of Graph theory. The first option is to use vertex graph coloring.

Vertex graph coloring $G = (V(G), E(G))$ is called the projection $c : V(G) \rightarrow S, S \subseteq N$ in which for all nodes *u* in graph

$\{u,v\} \in E(G) \Rightarrow c(u) \neq c(v)$ is valid. Elements of S set are called colors. Analogously, edge coloring can also be defined as that which will be continuously used. A coloring using at most k of different colors is called k-coloring. Graph G is k-colorable if there is s-coloring of graph G for $s \leq k$.

A Sudoku problem can be solved as a graph coloring problem when we try to color the graph using nine colors assuming that some of these colors are already used. The vertices of the graph will correspond with individual squares and the edges of the graph will link vertices that correspond to squares of one group.

Graf $G = (V(G), E(G))$ corresponding to the classic 9 × 9 Sudoku assignment will thus have 81 vertices that can be described as organized pairs of numbers 1 - 9.

$$V(G) = C \times C \text{ where } C = \{1, 2, \ldots, 9\}$$

Set of edges can be described as:

$$E(G) = \left\{ \left[[x; y]; [x'; y'] \right]; x = x' \vee y = y' \vee \left(\left\lceil \frac{x}{3} \right\rceil = \left\lceil \frac{x'}{3} \right\rceil \wedge \left\lceil \frac{y}{3} \right\rceil = \left\lceil \frac{y'}{3} \right\rceil \right) \right\},$$

Where $\lceil\ \rceil$ refers to the whole to part.

In this graph each vertex has degree 20, thus the number of edges is: $|\mathrm{H}| = \frac{20.81}{2} = 810$.

Solving the common 9 × 9 task this way is certainly possible but it is too lengthy. The graph coloring issue, therefore, is presented on a simpler version of Sudoku 4 × 4 (figure 3) for which the corresponding graph is constructed (figure 4).

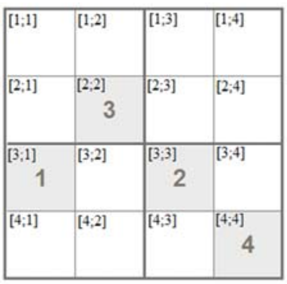

Figure 3: Sample version of 4 × 4 Sudoku

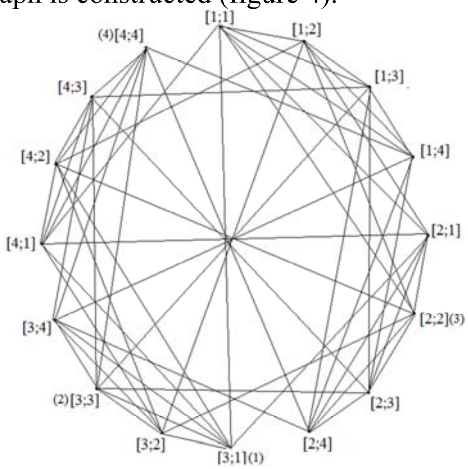

Figure 4: Graph corresponding to 4 × 4 Sudoku task

The above mentioned graph has 16 vertices and 56 edges. The vertices of this graph are numbered the same way as the individual squares in Sudoku assignment. Numerals in square brackets refer to the numbers that were already filled in the Sudoku.

For color labeling the numbers are continuously used depending on how the individual squares are pre-filled and a general heuristic algorithm is used.

General heuristic algorithm for graph coloring

1. In the graph the vertices that are not assigned to any color are found. If there is no color assigned to any vertex, then the algorithm ends.
2. From the set of vertices found on the basis of the first rule, the one that is adjacent to the maximal number of already colored vertices is chosen. When all vertices are colored, the solution is found and the algorithm ended.
3. From the set of vertices chosen with the second rule, the one that is adjacent to the largest number of uncolored vertices is chosen and colored by the color with the lowest value which was not used on its neighbors. If there are more such vertices one of them is selected randomly. In the next step the whole procedure is repeated from the very beginning. When all vertices are colored, the solution is found and the algorithm ended.

In this way, the position [3; 4] in the graph would be assigned color (3) because this vertex neighbors all the other colors. Subgraph[4] [3;1] [3;2] [3;3] and [3;4] is a complete graph and thus the vertex with coordinates [3;2] must be colored with color (4). In this way, the entire graph can be colored.

In the case of a general[5] graph, two problems can appear:

- This procedure cannot be applied because coloring of such a graph by four colors does not exist.
- The coloring exists but it cannot be found with this algorithm.

Such a method of manual processing is quite laborious and unclear because even in the simple assignment it is not visible enough which vertex is linked with all the other colored vertices. For this reason this graph coloring technique was partially changed and named Step after step.

Step after step method

This algorithm of graph coloring describes the ordinary way of thinking while solving Sudoku. The vertex graph coloring is combined with the edge graph coloring. Vertices of the graph are placed in the plane the same way the squares are placed in Sudoku. The edges of the complete subgraphs corresponding to individual groups are not plotted (the edges corresponding to pairs of vertices that cannot be colored with the same color are not plotted). It is, however, assumed that the solver is aware of which vertices are connected by edges. The whole algorithm can be divided into the following steps:

1. The vertex that is already colored is selected and linked by edges of same color with all other vertices of sets in which the vertex is located. These vertices can no longer be colored with the same color. This is repeated for all the vertices for which hints are given[6].

---

[4] A subgraph of a graph *G* is a graph whose vertex set is a subset of that of *G*, and whose adjacency relation is a subset of that of *G* restricted to this subset. In the other direction, a supergraph of a graph *G* is a graph of which *G* is a subgraph.

[5] In a mathematician's terminology, a graph is a collection of points and lines connecting some (possibly empty) subset of them.

[6] **Hint** is a pre-filled number in game assignment.

2. The vertices where the largest number of colored edges converge are found (it is most likely that there will be only one candidate[7]).
3. If there are vertices among them that can be colored only by one color, then they are colored with it and the procedure continues from the first step (there is no need to draw those edges into the graph that lead to a vertex where there is already another edge of the same color). If there are no such vertices, the procedure continues with the fourth step.
4. From the set of those selected vertices, the one that is adjacent to the largest number of uncolored vertices is chosen and colored to the color with the lowest value that is not used for its neighbors. If there are more such vertices one of them is selected randomly. In the next step the procedure continues from the first step.

Graphic form of Sudoku assignment (figure 3) is shown in figure 5. These edges that correspond to the vertices from the assignment were added (step 1).

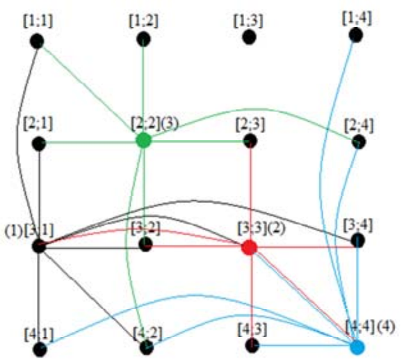

Figure 5: Sudoku graph after step one

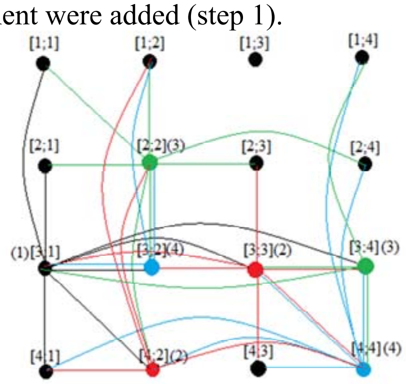

Figure 6: Sudoku graph after step two

In step two (also figure 5) three vertices can be found where lines of three colors converge and it is clear that this vertex must be colored with the last fourth color. Vertex [4;2] must be colored with color (2), vertex [3;4] with color (3) and vertex [3;2] with color (4). In step 3 the given vertices will be colored and relevant edges drawn (figure 6). If, for instance, vertex [4;2] is given color (2) it will not be lined by edge of this color with vertices [3;1] and [3;2]. Similarly, this would proceed with all the other vertices. The procedure will be repeated from step 1.

Based on the two above mentioned methods, it is possible to present issues related to Graph theory in an easy way.

In order to implement this strategy into a computer the matrix representation is used (table 1). In this table the first column lists all vertices of the graph together with the given colors. In each row there is a list of all neighbors of the vertex. This table corresponds to the solution of Sudoku by Step after step method. Similarly to the previous case it is clear that the lines corresponding to vertices [3;2], [3;4] and [4;2] are colored with three different colors and the vertices will, thus, be colored

[7] **Candidate** is a integer from the range 1 to 9 that can be added into an empty square. In this case, the candidate is a number from the range 1 to 4.

with the last, fourth color. The procedure then continues the same as in Step after step method.

List of vertices	Neighboring vertices						
[1;1]	[1;2]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[3;1] (1)	[4;1]
[1;2]	[1;1]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[3;2]	[4;2]
[1;3]	[1;1]	[1;2]	[1;4]	[2;3]	[2;4]	[3;3] (2)	[4;3]
[1;4]	[1;1]	[1;2]	[1;3]	[2;3]	[2;4]	[3;4]	[4;4] (4)
[2;1]	[1;1]	[1;2]	[2;2] (3)	[2;3]	[2;4]	[3;1] (1)	[4;1]
[2;2] (3)	[1;1]	[1;2]	[2;1]	[2;3]	[2;4]	[3;2]	[4;2]
[2;3]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[2;4]	[3;3] (2)	[4;3]
[2;4]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[2;3]	[3;4]	[4;4] (4)
[3;1] (1)	[1;1]	[2;1]	[3;2]	[3;3] (2)	[3;4]	[4;1]	[4;2]
[3;2]	[1;2]	[2;2] (3)	[3;1] (1)	[3;3] (2)	[3;4]	[4;1]	[4;2]
[3;3] (2)	[1;3]	[2;3]	[3;1] (1)	[3;2]	[3;4]	[4;3]	[4;4] (4)
[3;4]	[1;4]	[2;4]	[3;1] (1)	[3;2]	[3;3] (2)	[4;3]	[4;4] (4)
[4;1]	[1;1]	[2;1]	[3;1] (1)	[3;2]	[4;2]	[4;3]	[4;4] (4)
[4;2]	[1;2]	[2;2] (3)	[3;1] (1)	[3;2]	[4;1]	[4;3]	[4;4] (4)
[4;3]	[1;3]	[2;3]	[3;3] (2)	[3;4]	[4;1]	[4;2]	[4;4] (4)
[4;4] (4)	[1;4]	[2;4]	[3;4]	[4;1]	[4;2]	[4;3]	[3;3] (2)

Table 1: Matrix representation of Sudoku assignment

Another way of describing Sudoku solution is the use of independent sets.

Independent set

Set $A \subseteq V(G)$ is called an independent set of graph G if no two vertices of set A are linked by an edge. If it is possible to decompose the set of vertices of an acyclic[8] graph into two independent sets, the graph is called a bipartite[9] graph. If it is possible to decompose it into precisely k of independent sets, it is called a k-partite graph.

Each solution of Sudoku (9 × 9 version) can be described by nine independent subsets of the graph. Each of them has nine vertices labeled with the same number. Finding such a solution that would correspond to the task is, nevertheless, complicated.

Conclusion

This article concentrates on three methods of Sudoku solving based on use of Graph theory. Two of these methods were described in detail. Step after step method was created by the author of this article. Searching Sudoku solutions through Graph theory is considered an appropriate tool aimed to the clarification of the basic theoretical concepts in this field in the classroom. In a similar way, it is also possible to use Nim and base the solution search on the definition of the core of the graph and Sprague-Grundy theorem.

[8] An acyclic graph is a graph having no graph cycles. Acyclic graphs are bipartie .

[9] A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.

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ACTA MATHEMATICA 17

ON WEAK ISOMETRIES IN ABELIAN DIRECTED GROUPS

MILAN JASEM

ABSTRACT. *In the paper abelian directed groups with the basic intrinsic metric are assumed to be metric spaces and weak isometries, i. e. mappings preserving the basic intrinsic metric, are investigated. The main result is that for each weak isometry* f *in an abelian directed group* G *the relation* $f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$ *is valid for each* $x, y \in G$, *where* $U(x, y)$ *is the set of all upper bounds and* $L(x, y)$ *the set of all lower bounds of the set* $\{x, y\}$ *in* G. *This proposition generalizes a result of J. Rachůnek concerning isometries in 2-isolated abelian Riesz groups. Further, the notion of a subgroup symmetry is introduced and it is shown that subgroup symmetries and translations are two basic kinds of weak isometries in 2-isolated abelian directed groups and that each weak isometry in a 2-isolated abelian directed group is a composition of a subgroup symmetry and a translation.*

KEY WORDS: *directed group, intrinsic metric, weak isometry*

CLASSIFICATION: *Primary 06F15*

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Introduction

Isometries in abelian lattice ordered groups were introduced and studied by K. L. N. Swamy [10], [11]. J. Jakubík [1], [2] considered isometries in non-abelian lattice ordered groups. Isometries in abelian distributive multilattice groups were dealt with by M. Kolibiar and J. Jakubík in [4]. Weak isometries in lattice ordered groups were introduced by J. Jakubík [3]. J. Rachůnek [9] generalized the notion of an isometry to any po-group and investigated isometries in 2-isolated abelian Riesz groups. Weak isometries in some types of po-groups were studied in [5], [6].

Preliminaries

We review some notions and notations used in the paper.

Let G be a partially ordered group (po-group). The group operation will be written additively. If $A \subseteq G$, then we denote by $U(A)$ and $L(A)$ the set of all upper bounds and the set of all lower bounds of the set A in G, respectively. For $A = \{a_1, ..., a_n\}$ we shall write $U(a_1, ..., a_n)$ $(L(a_1, ..., a_n))$ instead of $U(\{(a_1, ..., a_n\})$ $(L(\{(a_1, ..., a_n\})$, respectively). If for $a, b \in G$ there exists the least upper (greatest lower) bound of the set $\{a, b\}$ in G, then it will be denoted by $a \vee b$ ($a \wedge b$, respectively).

The set of all subsets of a po-group G will be denoted by $exp\ G$.

A po-group G is called directed if $U(a, b) \neq \emptyset$ and $L(a, b) \neq \emptyset$ for each $a, b \in G$.

A po-group G is called 2-isolated if $2a \geq 0$ implies $a \geq 0$ for each $a \in G$.

A mapping g of a po-group G into G is called an involutory mapping (or an involution) if $g(g(x)) = x$ for each $x \in G$. We will write $g^2(x)$ instead of $g(g(x))$.

The absolute value $|x|$ of an element x of a po-group G is defined by $|x| = U(x, -x)$. If a po-group G is a lattice ordered group, then usually $|x| = x \vee (-x)$ for each $x \in G$.

Recall that if for elements x and y of a po-group G there exists $x \vee y$ in G, then there also exist $x \wedge y$, $-x \vee -y$, $-x \wedge -y$ in G and $(x \vee y) + c = (x + c) \vee (y + c)$, $c + (x \vee y) = (c + x) \vee (c + y)$ for each $c \in G$. Moreover, $-(x \vee y) = (-x) \wedge (-y)$. The dual assertions are valid, too.

Weak isometries in abelian directed groups

Swamy [10] showed that in any abelian lattice ordered group H the mapping

$$d\colon H \times H \to H \text{ defined by } d(x, y) = |x - y|$$

is an intrinsic metric (or an autometrization) in H, i. e. satisfies the formal properties of a distance function:

(M_1) $d(x, y) \geq 0$ with equality if and only if $x = y$ (positive definiteness),
(M_2) $d(x, y) = d(y, x)$ (symmetry),
(M_3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

J. Rachůnek [9] generalized the notion of an intrinsic metric for any po-group. He defined an intrinsic metric in a po-group G as a mapping $d\colon G \times G \to exp\, G$ satisfying the following conditions for each $x, y, z \in G$:

($\underline{M}_1$) $d(x, y) \subseteq U(0)$ and $d(x, y) = U(0)$ if and only if $x = y$,
($\underline{M}_2$) $d(x, y) = d(y, x)$,
($\underline{M}_3$) $d(x, y) \supseteq d(x, z) + d(z, y)$

and showed that in any 2-isolated abelian Riesz group H the mapping $d\colon H \times H \to exp\, H$ defined by $d(x, y) = |x - y|$ is an intrinsic metric in H.

In [7] it was shown that $d(x, y) = |x - y|$ is an intrisic metric in any 2-isolated abelian po-group. This intrinsic metric is called a basic intrinsic metric.

An important notion in the study of structures with a metric is a metric preserving mapping.

If G is a po-group, then a mapping $f\colon G \to G$ is called a weak isometry in G if

$$|f(x) - f(y)| = |x - y| \quad \text{for each } x, y \in G.$$

Hence, a weak isometry in a 2-isolated abelian po-group is a mapping which preserves the basic intrinsic metric $d(x, y) = |x - y|$.

A weak isometry f is called a stable weak isometry if $f(0) = 0$.

A weak isometry f is called an isometry if f is a bijection.

In [5] it was proved that any weak isometry in a directed group is a bijection. Hence the notions of an isometry and of a weak isometry coincide in any directed group.

If G is a po-group and $d \in G$, then a mapping $f\colon G \to G$ such that $f(x) = x + d$ for each $x \in G$ is called a translation in G.

This definition of a translation is in close analogy with definition of a translation in the Euclidean plane R^2.

The identity mapping can be considered as a special translation with $d = 0$. Every translation in a po-group is an isometry.

If f is a weak isometry in a po-group G, then the mapping g defined by $g(x) = f(x) - f(0)$ for each $x \in G$ is a stable weak isometry in G. Hence $f(x) = g(x) + f(0)$ for each $x \in G$. If we put $h(x) = x + f(0)$ for each $x \in G$, then $f(x) = h(g(x))$ for each $x \in G$.

Therefore each weak isometry in a po-group is a composition of a stable weak isometry and a translation.

In [9] J. Rachůnek proved that for any isometry f in a 2-isolated abelian Riesz groups G the following condition

$$f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$$

is satisfied for each $x, y \in G$ (Theorem 2.2).

We will extend this proposition to all abelian directed groups.

Theorem 1. Let G be an abelian directed group and g a weak isometry in G. Let $x, y \in G$, $x \geq y$. Then $g(x) + x \geq g(y) + y$, $-g(x) + x \geq -g(y) + y$.

Proof: Let $x, y \in G$, $x \geq y$. Then from $|x - y| = |g(x) - g(y)|$ we get $U(x - y) = U(x - y, y - x) = U(g(x) - g(y), g(y) - g(x))$. This implies $x - y = (g(x) - g(y)) \vee (g(y) - g(x))$. Thus $x - y \geq g(x) - g(y)$, $x - y \geq g(y) - g(x)$. Hence $-g(x) + x \geq -g(y) + y$, $g(x) + x \geq g(y) + y$.

As a consequence of Theorem 1 we obtain:

Corollary 1. If G is an abelian directed group, g a stable weak isometry in G and $x \in G$, $x \geq 0$, then $g(x) + x \geq 0$, $-g(x) + x \geq 0$.

Theorem 2. Let G be an abelian directed group and f a weak isometry in G. Let $x, y \in G$, $y \leq x$. Then $f([y, x]) = [y - x + f(y), x - y + f(y)] \cap [y - x + f(x), x - y + f(x)]$.

Proof: Let $x, y \in G$, $y \leq x$. Let $a \in [y, x]$. Thus $y \leq a \leq x$. This yields $f(a) + y \leq f(a) + a \leq f(a) + x$, $-f(a) + y \leq -f(a) + a \leq -f(a) + x$.

From Theorem 1 it follows that $f(y) + y \leq f(a) + a$. Since $f(a) + a \leq f(a) + x$, we have $f(y) + y \leq f(a) + x$. Hence $y - x + f(y) \leq f(a)$.

According to Theorem 1, $-f(y) + y \leq -f(a) + a$. From this and $-f(a) + a \leq -f(a) + x$ we get $-f(y) + y \leq -f(a) + x$. Thus $f(a) \leq x - y + f(y)$.

By Theorem 1, $-f(a) + a \leq -f(x) + x$. Since $-f(a) + y \leq -f(a) + a$, we have $-f(a) + y \leq -f(x) + x$. Then $y - x + f(x) \leq f(a)$.

In view of Theorem 1 we have $f(a) + a \leq f(x) + x$. This and $f(a) + y \leq f(a) + a$ yields $f(a) + y \leq f(x) + x$. Thus $f(a) \leq x - y + f(x)$.

Therefore $f([y, x]) \subseteq [y - x + f(y), x - y + f(y)] \cap [y - x + f(x), x - y + f(x)]$.

Let $b \in [y - x + f(y), x - y + f(y)] \cap [y - x + f(x), x - y + f(x)]$. Since each weak isometry in directed group is a bijection, there exists $c \in G$ such that $f(c) = b$. Then $y - x + f(y) \leq f(c) \leq x - y + f(y)$ and hence $f(y) - f(c) \leq x - y$, $f(c) - f(y) \leq x - y$.

Then we obtain $x - y \in U(f(y) - f(c), f(c) - f(y)) = |f(c) - f(y)| = |c - y| = U(c - y, y - c)$. Thus $x - y \geq c - y$. This implies $x \geq c$.

Since $y - x + f(x) \leq f(c) \leq x - y + f(x)$, we have $f(x) - f(c) \leq x - y$, $f(c) - f(x) \leq x - y$. Hence $x - y \in U(f(x) - f(c), f(c) - f(x)) = |f(c) - f(x)| = |c - x| = U(c - x, x - c)$. Thus $x - y \geq x - c$. This yields $c \geq y$. Thus $c \in [x, y]$ and hence $b = f(c) \in f([x, y])$. Therefore $[y - x + f(y), x - y + f(y)] \cap [y - x + f(x), x - y + f(x)] \subseteq f([y, x])$.

Theorem 3. Let G be an abelian directed group and f a weak isometry in G. Then the following condition.

(C) $\quad f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$

is satisfied for each $x, y \in G$.

Proof: Each weak isometry in G is a composition of a stable weak isometry and a translation. By Theorem 2.7 [9], any translation in a po-group satisfies the condition (C). Thus it suffices to consider that f is a stable weak isometry in G.

Let $x, y \in G$, $a \in U(L(x, y)) \cap L(U(x, y))$. Let $u \in L(f(x), f(y))$, $v \in U(f(x), f(y))$. Thus $f(x), f(y) \in [u, v]$.

By Theorem 3 [5], each stable weak isometry in a directed group is an involutory group automorphism. Then in view of Theorem 2 we have $u - v + f(u) \le f^2(x) = x \le v - u + f(u)$, $u - v + f(v) \le f^2(x) = x \le v - u + f(v)$, $u - v + f(u) \le f^2(y) = y \le v - u + f(u)$, $u - v + f(v) \le f^2(y) = y \le v - u + f(v)$. Hence $v - u + f(u)$, $v - u + f(v) \in U(x, y)$, $u - v + f(u)$, $u - v + f(v) \in L(x, y)$. From this follows that $u - v + f(u) \le a \le v - u + f(u)$, $u - v + f(v) \le a \le v - u + f(v)$. Hence $a \in [u - v + f(u), v - u + f(u)] \cap [u - v + f(v), v - u + f(v)]$. Since $a = f^2(a)$, from Theorem 2 it follows that $f(a) \in [u, v]$. Since u was an arbitrary element of $L(x, y)$ and v an arbitrary element of $U(x, y)$, we have $f(a) \in U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$.

Therefore $f(U(L(x, y)) \cap L(U(x, y))) \subseteq U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$.

Then for $x_1 = f(x)$, $y_1 = f(y)$ we get $f(U(L(x_1, y_1)) \cap L(U(x_1, y_1))) \subseteq U(L(f(x_1), f(y_1))) \cap L(U(f(x_1), f(y_1)))$. Since f is an involutory bijection, we have $U(L(x_1, y_1)) \cap L(U(x_1, y_1)) = f^2(U(L(x_1, y_1)) \cap L(U(x_1, y_1))) \subseteq f(U(L(f(x_1), f(y_1))) \cap L(U(f(x_1), f(y_1)))) = f(U(L(f^2(x), f^2(y))) \cap L(U(f^2(x), f^2(y)))) = f(U(L(x, y)) \cap L(U(x, y)))$. Hence $U(L(f(x), f(y))) \cap L(U(f(x), f(y))) \subseteq f(U(L(x, y)) \cap L(U(x, y)))$. This ends the proof.

Corollary 2. Let G be an abelian directed group, f a weak isometry in G, $x, y \in G$. If there exist $x \wedge y$, $x \vee y$, $f(x) \wedge f(y)$ and $f(x) \vee f(y)$ in G, then $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

Proof: If there exist $x \wedge y$, $x \vee y$, $f(x) \wedge f(y)$ and $f(x) \vee f(y)$ in G, then $U(L(x, y)) \cap L(U(x, y)) = [x \wedge y, x \vee y]$, $U(L(f(x), f(y))) \cap L(U(f(x), f(y))) = [f(x) \wedge f(y), f(x) \vee f(y)]$. Then from Theorem 3 it follows that $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

From Corollary 2 we immediately obtain the following assertions.

Corollary 3. Let G be an abelian directed group, f a weak isometry in G, $x, y \in G$, $x \le y$.

(i) If $f(x) \le f(y)$, then $f([x, y]) = [f(x), f(y)]$.
(ii) If $f(y) \le f(x)$, then $f([x, y]) = [f(y), f(x)]$.
(iii) If there $f(x) \wedge f(y)$ and $f(x) \vee f(y)$ exist in G, then $f([x, y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

Since any Riesz group is a directed group, Theorem 3 generalizes Theorem 2.2 of J. Rachůnek [9].

Elements x and y of a lattice ordered group are called orthogonal if $|x| \wedge |y| = 0$.

J. Rachůnek [12] introduced the following notion of disjointness in po-groups as a generalization of the notion of orthogonality in lattice ordered groups.

Elements x and y of a po-group H are called disjoint (notation $x\delta y$) if there exist $x_1 \in |x|$ and $y_1 \in |y|$ such that $x_1 \wedge y_1 = 0$.

In close analogy with the definition of an axial symmetry in the Euclidean plane R^2 we introduce the following notion of a subgroup symmetry.

Let G be a po-group, A a subgroup of G and d the basic intrinsic metric in G. A mapping $f: G \to G$ is called a symmetry with the respect to subgroup A if the following conditions are satisfied for each $x \in A$, $y \in G \setminus A$:

(i) $f(x) = x$, $f(y) \neq y$,

(ii) $d(f(y), x) = d(y, x)$,
(iii) $(y - f(y))\delta x$.

Theorem 4. Let G be an abelian directed group and f a stable weak isometry in G. Let $x, y \in G$, $x, y \geq 0$. Then $(-f(x) + x) \wedge (f(y) + y) = 0$.
Proof: Let $z \in G$, $z \geq 0$. Then from $|z - 0| = |f(z) - f(0)|$ it follows that $z = (-f(z)) \vee f(z)$. This implies $(-f(z)) \wedge f(z) = -z$. From this we obtain $(-f(z) + z) \wedge (f(z) + z) = 0$.

Let $x, y \in G, x, y \geq 0, v \in U(x, y)$. Then from Theorem 1 it follows that $0 = -f(0) + 0 \leq -f(x) + x \leq -f(v) + v$, $0 = f(0) + 0 \leq f(y) + y \leq f(v) + v$. Since $(-f(v) + v) \wedge (f(v) + v) = 0$, we have $(-f(x) + x) \wedge (f(y) + y) = 0$.

Theorem 5. Let G be a 2-isolated abelian directed group and f a stable weak isometry in G. Let $A = \{ x \in G;\ f(x) = x\}$, $B = \{x \in G;\ f(x) = -x\}$. Then A, B are directed convex subgroups of G.
Proof. Let $x, y \in A, z \in G, x \leq z \leq y$. Thus $f(x) = x, f(y) = y$. By Corollary 3(i), $f([x, y]) = [x, y]$. Hence $x \leq f(z) \leq y$. Then from $U(y - z) = |y - z| = |f(y) - f(z)| = |y - f(z)| = U(y - f(z))$ it follows that $y - z = y - f(z)$. This implies $f(z) = z$. Hence A is a convex subset of G.

Let $a, b \in A$. Since f is a group homomorphism, we have $f(a - b) = f(a) - f(b) = a - b$. Hence $a - b \in A$. Therefore A is a subgroup of G.

Further, G is a directed group and hence there exists $u \in G$, such that $u \geq a - b, u \geq 0$. Since f is an involutory group homomorphism, we have $f(u + f(u)) = f(u) + f^2(u) = f(u) + u$. Thus $u + f(u) \in A$. In view of Theorem 1 and Corollary 1 we obtain $f(u) + u \geq f(a - b) + a - b$, $f(u) + u \geq 0$. Then $2(f(u) + u) \geq f(a - b) + a - b = 2(a - b)$. Since group G is 2-isolated, we get $f(u) + u \geq a - b$. Then $f(u) + u + b \in A, f(u) + u + b \geq a, f(u) + u + b \geq b$. Thus A is a directed subgroup of G.

Let $x, y \in B, z \in G, x \leq z \leq y$. Since $f(x) = - x, f(y) = - y$, in view of Corollary 2(i) we obtain $f([x, y]) = [-y, -x]$. Hence $-y \leq f(z) \leq -x$. Then from $U(y - z) = |y - z| = |f(y) - f(z)| = |-y - f(z)| = U(f(z) + y)$ we get $y - z = f(z) + y$. This yields $f(z) = -z$. Hence B is a convex subset of G.

Let $c, d \in B$. Since f is a group homomorphism, we have $f(c - d) = f(c) - f(d) = -c + d = -(c - d)$. This implies $c - d \in B$. Therefore B is a subgroup of G.

Since G is a directed group, there exists $v \in G$, such that $v \geq c - d$, $v \geq 0$. Since f is an involutory group homomorphism, we have $f(v - f(v)) = f(v) - f^2(v)) = f(v) - v = -(v - f(v))$. Thus $v - f(v) \in B$. By Theorem 1 and Corollary 1, $-f(v) + v \geq -f(c - d) + c - \mathrm{d}$, $-f(v) + v \geq 0$. Then $2(-f(v) + v) \geq -f(c - d) + c - d = 2(c - d)$. Since group G is 2-isolated, we have $-f(v) + v \geq c - d$. Then $-f(v) + v + d \in B$, $-f(v) + v + d \geq c, -f(v) + v + d \geq d$. Hence B is a directed subgroup of G.

Theorem 6. Each stable weak isometry in a 2-isolated abelian directed group G is a subgroup symmetry.
Proof. Let f be a stable weak isometry in G. Let $A = \{x \in G; f(x) = x\}$, $B = \{x \in G; f(x) = -x\}$. We will show that f is a symmetry with respect to A. Clearly $f(y) \neq y$ for each $y \in G\backslash A$. Let $y \in G\backslash A$, $x \in A$. Then $d(y, x) = d(f(y), f(x)) = d(f(y), x)$. Further, we have $f(y - f(y)) = f(y) - f^2(y) = f(y) - y = -(y - f(y))$. Thus $y - f(y) \in B$. Analogously $f(y) - y \in B$. By Theorem 5, A and B are directed groups. Hence there exist $u \in B$, $v \in A$ such that $u \in U(y - f(y), f(y) - y, 0)$, $v \in U(x, -x, 0)$.

By Theorem 4, $(u - f(u)) \wedge (v + f(v)) = 0$. Thus $(2u) \wedge (2v) = 0$ and hence $u \wedge v = 0$. This yields $(y - f(y))\delta x$. Therefore f is a symmetry with respect to subgroup A.

Corollary 4. Each weak isometry in a 2-isolated abelian directed group is a composition of a subgroup symmetry and a translation**.**

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FACULTY OF NATURAL SCIENCES
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THE ROLE OF THE GRAPHIC DISPLAY CALCULATOR IN FORMING CONJECTURES ON THE BASIS OF A SPECIAL KIND OF SYSTEMS OF LINEAR EQUATIONS

JOANNA JURECZKO

ABSTRACT. *A graphic display calculator (GDC) was introduced to mathematical education in the 90s' of the last century. Since then a great deal of scientists and teachers have suggested that this portable device could be applied effectively in the process of teaching and learning mathematics. The aim of this paper is to analyze the process of forming conjectures on the base of some special systems of linear equations in respect of the usage of technology. The researched group consisted of students between the age of 17 and 19, who used GDC as a mandatory device during learning mathematics. The results will be compared with some presented in the paper [1] where one can find different kinds of GDC applications in the process of learning mathematics and the process of generalization with GDC usage analyzed in [4] where visual template tasks were taken into consideration.*

KEYWORDS: *graphic display calculator, mathematics learning, forming conjectures, generalization, International Baccalaureate Diploma Programme*

CLASSIFICATION: *B10, C70, D40, H10, H30.*

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Introduction

The introduction of a graphic display calculator (GDC) into mathematics education began in the early 90s' of the last century. Since then a lot of scientists have examined and evaluated the role of GDC in teaching and learning mathematics as far as different types of mathematical activities are concerned. Throughout years GDC has become an obligatory tool for each student in particular educational programmes. Among them there is International Baccalaureate Diploma Programme which is designed for students aged 16-19. What is worth noticing the International Baccalaureate Organization (IBO), founded in the 1960s', is now a leader in the world education[1]. Moreover, the programme is suitable for conducting the research about using GDC in some aspects of learning mathematics. A lot of the researchers proposed the classification of GDC usage in different activities. For example in paper [1] authors try to answer the question "How do students use GDC to support their mathematics education?" Furthermore, they consider some limitations and constraints of using GDC technology that emerged within the classroom and homework practice. In this paper [1] (p. 151) one can find some patterns and modes of the graphic calculator use.

[1] More information about this programme one can get on www.ibo.org or [5], [7] and [8] .

To quote:

Role of the Graphic Calculator	Description of students Actions
Computational Tool	evaluating a numeral expression, estimating and rounding
Transformational Tool	changing the nature of the task
Data Collection and Analysis Tool	gathering data, controlling phenomena, finding patterns
Visualizing Tool	finding symbolic functions, displaying data, interpreting data, solving equations
Checking Tool	confirming conjectures, understanding multiple symbolic forms

Table 1. Classification of using GDC in different aspects proposed in [DZ].

In paper [4] the process of generalization was considered, with respect of using so called visual patterns, in which on the basis of a prior experience and research, I proposed the scheme of generalization using graphic display calculator.

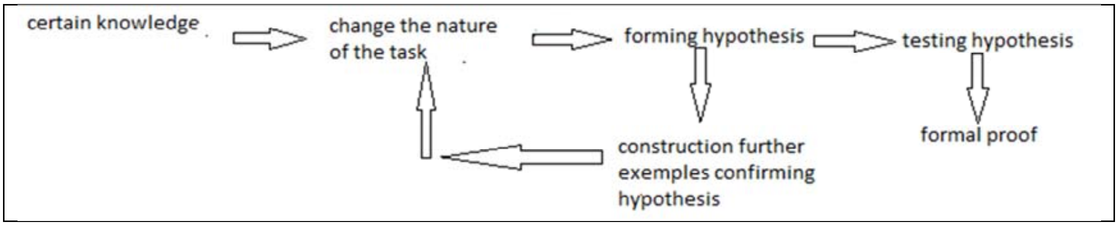

Table 2. The scheme of process of generalization proposed in [4]

The question for the paper is

1. How do the students reason and form conjectures on a basis of a special kind of systems of linear equations?
2. How can the graphic display calculator help students in forming conjectures?
3. Is the process of forming conjectures in this research similar to the process of generalization proposed in [4]?

Methodology and data analysis

In the research students from two different groups of IB class were taken into account. Additionally, all students were taught by me. Furthermore, every student attended the International Baccalaureate Diploma Programme class and all of them started using GDC about 8 months beforehand the research. Prior to the research students were taught about different methods of solving systems of equations (without using any technologies) and they had knowledge about an arithmetic sequence. On the occasion of teaching sequences, students became familiarized with forming conjectures. However, they did not have any experience with working with such exercises as were proposed during the research. In my diagram the described step is called "certain knowledge".

The data collection was completed in two separated groups of students.

In the first 5-student-group I gave the task[2] and discussed the problem. However, no exact instructions for solving it were provided. During 10-day-period students solved this task individually as their homework without any assistance (with access to GDC). After 10 days students submitted the final version of their solutions. In the second 10-student-group the students were given the task and ordered to solve it during normal lesson time (three consecutive 45-minute-lessons in one day). Similarly as before students did not obtain any special instructions of how to solve the task. During the whole time students worked individually using only sheets of paper, pens and GDCs. After the whole process I interviewed the students whether GDC helped them solve the task. However, only in second group students answered the question in written way. As a teacher I knew the limitations of GDC which could disturb solving this task. Yet I did not inform the students about them. The text of the task given to students is provided below.

Let us consider the 2x2 system of the equation $$\begin{cases} x + 2y = 3 \\ 2x - y = -4 \end{cases}$$ Examine the constants in both equations. Solve the system. Create and solve a few more similar 2x2 systems. Make a conjecture and prove it. Extend your investigation to 3x3 systems. Make a conjecture and prove it.

Table 3. The text of the first part of the task used in the research.

This task was not chosen by accident, because a question formulated in such a way can be considered as an open problem which might seem interesting for students (especially for using GDC) and creates an opportunity for experiment and generalization, which is one of the most important purpose of teaching mathematics. Moreover, it enables the observation of different approaches to solving the same problem.

Analysis of students' work

The first question was considerably easy to answer for students. Most of them did not use GDC, and solved it using method of elimination. After analyzing the constants in this systems all students very quickly noticed that coordinates made an arithmetic sequence with different common difference in each equation[3]. They created similar systems using distinct common difference in both equations. Moreover, they quickly realized that all their examples gave the same solutions x=-1 and y=2. For solving further similar systems of equations all students used GDC (mode: EQUA) because, as they commented, not only did they wish to solve them very quickly they also did not want to make any mistakes. As we can see above, students considered systems with distinct common differences in order to check if they would obtain the same result

$\begin{cases} x + 4y = 7 \\ 4x + 2y = 0 \end{cases}$	$\begin{cases} -5x - y = 3 \\ 6x + 5y = 4 \end{cases}$	$\begin{cases} 4x + 10y = 16 \\ 12x + 7y = 2 \end{cases}$	$\begin{cases} 2x + 4y = 6 \\ 4x - 2y = -8 \end{cases}$	$\begin{cases} \frac{1}{2}x + 2y = \frac{7}{2} \\ -2x - \frac{5}{2}y = -3 \end{cases}$

Table 4. Some examples of similar systems proposed by students

2 This task was taken as a part of Portfolio – Internal Assessment for International Baccalaureate Diploma Programme in 2011-2012. In [2] one can find similar system but with different questions. This research was carried out independently.

3 In paper [2] author observed that although their students knew nothing about arithmetic sequence they came to the same conclusion.

In this part students generally used GDC for solving their examples, as they claimed in order to avoid mistakes in calculations and to examine further examples. However, some students preferred methods of elimination instead of using GDC. In my scheme this step is called “change of the nature of the task” .

The next point was concerned with making a conjecture with proof. However, it appeared to be too difficult for students. Only four students from the first group made conjecture properly and proved it. Some of students’ propositions are given below[4]

> Let us assume that a is the first term and d is the common difference of the first equation and that b is the first term and kd is the common difference of the second equation.
>
> $$\begin{cases} ax + (a+d)y = a + 2d \\ bx + (b+kd)y = b + 2kd \end{cases}$$

The student did not explain why she used such common differences (where the second one was the multiplication of the first one). Another student proposed a different general pattern for the system.

$$\begin{cases} ax + (a+d)y = a + 2d \\ bx + (b+f)y = b + 2f \end{cases}$$

The third student proposed the following general pattern.

$$\begin{cases} ax + (a+1)y = (a+2) \\ bx + (b-3)y = (b-6) \end{cases}$$

Students at this stage solved their general systems using method of elimination or Cramer’s rule.

It is crucial to notice that each student proposed different general pattern for the system and no one assumed anything in respect to constants. In the first example student used multiplication of the common difference, in the second equation, but in the third example the student used fixed common difference. Only in the second example the generalization was done properly. Nevertheless, nothing was assumed about constants. This step is called “testing hypothesis”. It is worth noticing that students did not do this step well and after forming conjectures they omitted the step “construction of further examples confirming hypothesis”.

In the second group, which worked during the lesson time none of the students generalized this system and no one proposed a formal proof. As they claimed they did not have enough time to do it and they were more focused on producing as many examples as possible to form a conjecture. Similarly, also in this group the step “construction further examples confirming hypothesis” was omitted

The next question of the task concentrated on further examination of systems of type 3x3. Students tried to solve 3x3 systems in which coordinates formed different arithmetic sequences. However, they were disappointed because in each system GDC could not find a solution[5]. What is strange, in comparison to the previous part of the task, students did not use as many examples for forming conjectures. When they examined one or two examples

[4] In the boxes there are parts of original students’ work

[5] Casio model fx-9860GII which students used is not able to solve systems of no solutions and infinitely many solutions

they were quickly discouraged by this part. Although some students tried to solve particular examples using method of elimination, others (especially in the second group) used Cramer's rule where GDC was very helpful in counting needed determinants of matrices (students used RUN-MATH mode for this purpose). In the process of forming conjectures students from the first group used the method of row operations or Cramer's rule. Yet, the same students made the same mistakes in generalization of the system 3x3 as in the generalization of the system 2x2. Two examples of generalized systems are shown below

$\begin{cases} ax+(a+d)y+(a+2d)z=(a+3d) \\ bx+(b+f)y+(b+2f)z=(b+3f) \\ cx+(c+g)y+(c+2g)z=(c+3g) \end{cases}$	$\begin{cases} ax+(a+1)y+(a+2)z=(a+3) \\ bx+(b-3)y+(b-6)z=(b-9) \\ cx+(c+3)y+(c+6)z=(c+9) \end{cases}$

As we can see in the second example the same mistake occured as before (the common difference is fixed in each equation). In conclusion, none of the students from the second group gave general pattern for such a system.

Conclusions

In order to find out a similarity to paper [1] students in this task used GDC as:

- a computational tool (to calculate determinants of matrices in Cramer's rule),
- a transformational tool (to analyze similar systems of equations)
- a visualizing tool (solving systems of equations)

What is important to notice, students did not use the GDC as a checking tool, because as was mentioned before they omitted the step "construction of further examples confirming hypothesis."

To conclude, some problems were confirmed by the research, such as: using GDC in solving problems can encourage students to form conjectures and to produce their own tasks (in this research tasks were similar to proposed by the teacher). Using GDC allowed students to concentrate deeper on the task without worrying about mistakes which could appear during the traditional (paper-pencil) solving of particular systems. Moreover, the students could examine further similar systems in shorter time to observe similarities, what helped them form conjectures. As far as analyzed examples are concerned, students made some mistakes in the process of proving the task. One can conclude that this stage did not facilitate students' thinking or students had a problem with formal proofs. One can think that information about relations between constants in the systems in terms of arithmetic sequence was not important for students.[6]

Students using GDC can work almost as a researcher but GDC does not replace mathematical thinking and does not kill it. Some limitations of GDC (especially when students were not able to solve 3x3 systems which gave infinitely many solutions) show that this device is one of many but not the only device helpful in solving problems. Students noticed these limitations of using GDC but they felt a bit discouraged by using

[6] In paper [2] students did not know anything about arithmetic system but they could solved the problem properly (without generalizations and examining the 3x3 systems).

GDC in the first approach as they tried to find another mode of GDC which could help continue solving the task.

What is important to notice, students (working independently) omitted the step "construction further examples confirming their hypothesis". They formed hypothesis only after the examination of a few examples. After forming the hypothesis they finished the part of the task or (students from the first group only) moved to the next part of the task – formal proof.

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ACTA MATHEMATICA 17

DEVELOPMENT OF GEOMETRIC IMAGINATION IN LOWER SECONDARY EDUCATION

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ABSTRACT. *This article approaches the importance and significance of spatial intelligence; moreover it defines areic, geometric imagination and spatial intelligence. Specifically the Tangram brain teaser is suggested as a tool to be used for further development of geometric imagination in lower grade pupils of secondary schools. Pupils shall achieve better understanding of geometry by learning to solve Tangram tasks.*

KEY WORDS: *geometric imagination, Tangram, activities*

CLASSIFICATION: *D43*

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Introduction

Imagination is one of the most important skills in a person's possession and it serves greatly in the versatile development of the pupil's personality. According to several psychologists a developed imagination is closely related to a life success. The imagination is also linked to coping strategies such as the ability to park your car, to place furniture in the room or to orientate in foreign city. Spatial imagination is formed with the help of the real world at an early age of an individual and it is developed along with the sense of the third dimension. The well-developed spatial imagination has a huge value in today's society. Some fields, such as sculpture or topology, would not even exist without developed spatial minds. It has also an important role in science itself where it can serve different purposes. Imagination can be a useful tool, an auxiliary way of thinking, a way to obtain information, a way to formulate tasks or a direct means to solve the problem. By the term of imagination we mean the ability to create images in one's mind, store them in memory and later recall them. [1]. According to [2] the core of spatial intelligence are the capabilities that ensure accurate perception of the visual world and help transform and modify the original perceptions and create own visual experience and conceptual imagination. With those capabilities, we can construct various shapes and manipulate with them. In this article we approach the importance and the significance of spatial intelligence; we define concepts such as areic and spatial imagination and intelligence and suggest ways to develop geometric imagination in lower secondary education through specific tasks, using the Tangram brain teaser.

Geometric imagination in mathematics

Imagination poses an important role in mathematics, and especially in geometry. The notion of geometric imagination is currently used as an ability to imagine:

- geometric figures, their size, position in space and their characteristics,
- given geometric figure in a different position than the original one,
- the change in shape, size, structure and other properties of the figure,
- shapes based on their verbal description,
- shapes by planar image. [3]

The geometric imagination is defined in [4] as the ability to recognize geometric figures and their characteristics, to abstract the geometric attributes of specific objects and to see in them the geometric shapes in pure form; to imagine geometric shapes based on planar patterns in a variety of mutual relations, even those that cannot be demonstrated using physical models. The author states that it is also important to have resources of geometric figures and consequently the ability to recall a variety of forms and shapes and finally the ability to imagine geometric figures and the relationships between them based on their description. The overall layout of graphics on paper, the perception of shape and area, and the relative positions of several plane figures are all related to the areic imagination [5]. Thus, developing areic imagination supports the development of spatial imagination. Spatial imagination is an important part of the child's cognitive competencies. According to [2] spatial ability is divided into three components: the ability to recognize the identity of an object observed from different angles; the ability to imagine movement or change in the internal arrangement of a particular configuration; the ability to think about spatial relations, which are dependent on the orientation of the body of the observer. Solving specific tasks which are aimed at the development of planar imagination can lead to achieving better results in geometry, and solving geometric problems develops spatial imagination. [6] It is important to begin with the development of imagination in early preschool age. There are two favorable periods in the life of a child for the development of imagination: ages between 5-6 years and age of 10-11 years. [7]

Tangram and its use in school curriculum

Brain teaser Tangram is inciting and interesting teaching material that can be used for developing pupils' spatial imagination, logical and creative thinking. It is a convenient combination of didactic game and teaching aid. The word "tangram" means any convex unit divided into seven parts, of which we can compose various figures. Tangram consists of five equilateral right triangles (two large, one medium and two in a smaller size), a square and a rhomboid. Activities with Tangram offer vast possibilities for applying of creative thinking, imagination and creativity. There are many methodological manuals dealing with various tasks and motivational activities with Tangram [1], [6], [8], [9]. According to the State educational program [10] for lower secondary education the theme 'Geometry and measurement' should in parallel develop both areic and spatial imagination and it should also extend pupils' knowledge of geometric figures. According to content and performance standards pupils should understand the characteristics of basic geometric figures, and be able to solve positional and metric problems of ordinary life. An important point here is the development of spatial imagination for which Tangram could be the ideal tool. For this purpose, we have proposed several activities with enhancing level of performance that contribute to the development of geometric and spatial imagination of pupils. We addressed the use of Tangram in the theme 'Area of plane figures' since for this theme there are very little activities, if any. Our suggested activities were carried out at selected schools in Bratislava in March 2014. There were 26 pupils aged 11-12 years (6th grade) in the experimental class. As it is already mentioned above, this period is one of the most suitable for the development of geometric imagination. We took advantage of four lessons of mathematics, which ran within one week. Each pupil drew Tangram in advance, clipped out different geometric figures and painted the figures random color, as we did not have original Tangram for every pupil.

Activities with Tangram for developing geometric imagination

At the beginning we have familiarized pupils with what actually Tangram is and what the basic rules of this puzzle are. Surprising was the finding that none of the pupils knew Tangram yet. In the first stage pupils have been reminded the geometrical figures of which Tangram is composed of and their characteristics. All pupils knew the triangle and square, but they weren't familiar with rhomboid yet. Therefore, we defined a new unit which we presented them as a parallelogram, in which adjacent sides are of unequal lengths and angles are non-right angled.

Figure 1: Pupils when doing Tangram activities

In the second stage, pupils were given the task to assemble the model according to the different shapes of animals, human figures and objects. This task was easily mastered by all the pupils in a relatively short time. Then they got task to put together various images, but in a draft received only the outer figure contour, so they did not know the inner subdivision of composite models (Fig. 2). By compiling these patterns child learns to see the area. Majority of pupils had with this activity problem and few pupils failed to compile the given figure even after several unsuccessful attempts and a sufficiently long time.

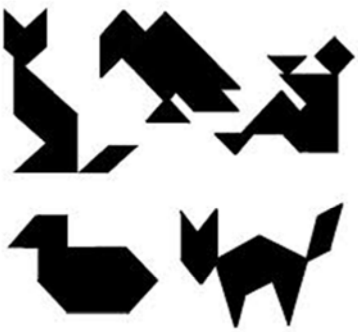

Figure 2: Figures only with outer contour

In the next activity, students were given the task to organize all parts of Tangram into any figure that reminds them of something - an image from ordinary life, which they should additionally name. This activity gives pupils room to apply imagination, originality and creativity and encourages pupils to be active.


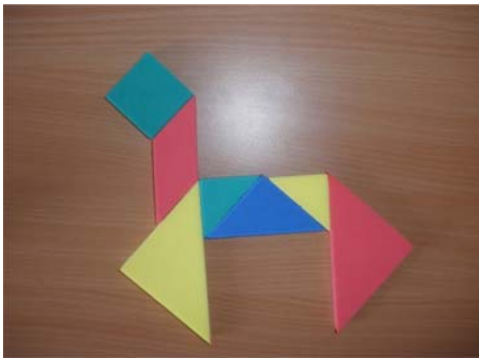

Figure 3: Objects created by pupils – Giraffe and Horse

Next phase we have devoted to activities aimed at the area of the geometrical figures. According to the State educational program [10] pupils should be able to calculate the perimeter and area of the rectangle and square, and also analyze the elements composed of squares and rectangles. Pupils should be also able to design their own methods for calculating the perimeter and the area of geometric figures composed of squares and rectangles. During the verification of these activities pupils have already known to calculate the area of square and rectangle, but they have not yet met with a triangle, which is thematically sorted into a higher grade. Therefore, in this activity we decided that the area of one of the smallest right triangles would be one area unit. Subsequently, we determined the area of the remaining parts of Tangram according to this unit. Some pupils immediately came to the knowledge that square will represent two units of area, since it consists of two identical smallest triangles. To determine area of rhomboid, most pupils had trouble to determine how many units compose its area. Pupils didn't know to combine two smallest triangles in such a way as to give the rhomboid. After some time we determined the area of all remaining parts.

Consequently, pupils should have determined what area has the whole Tangram puzzle. The best pupils of the class after a short while responded with 16 area units. Then they were given the task to build a square and a rectangle, using all parts of Tangram. After our question, which of these two figures has a bigger area, automatically a few pupils answered that rectangle. They absolutely didn't realize that figures composed of exactly the same parts must still have the same area.

In the next activity, pupils were supposed to put together their own figures composed of 12 area units, 14 area units and 15 area units. In this activity pupils should understand that not every figure which has the same area must also have the same shape. Here are a few pupils' works:

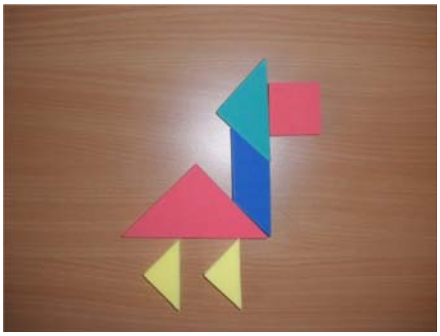
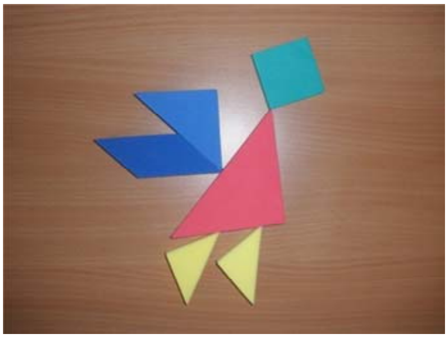

Figure 4: Camel and Cock (12 area units)

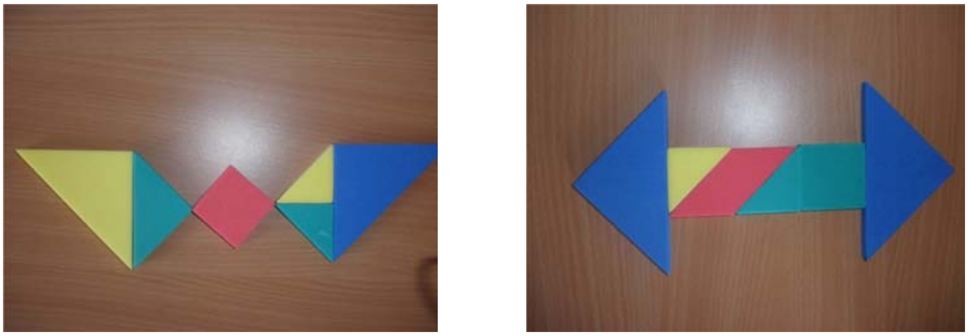

Figure 5: Bat and Arrow (14 area units)

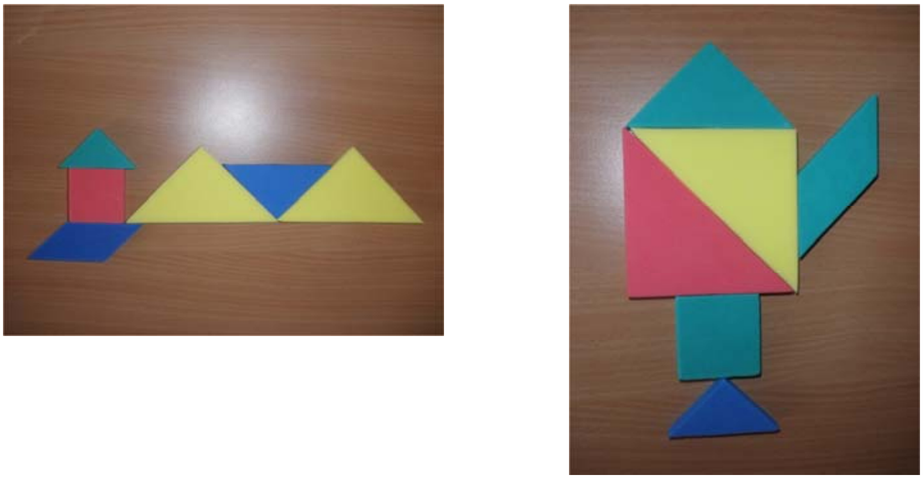

Figure 6: Lake house in the mountains – Fairy tale house (15 area units)

Conclusion

In this article, we have presented specific activities using the brain teaser Tangram, which develop geometric imagination of pupils. We found out that our proposed activities motivate pupils to use their imagination and provide them the opportunity to develop their creative and logical thinking. During the observation of pupils doing the proposed activities, we figured that pupils have lack practical experience with geometry and with changing positions of geometrical figures, which are some of the most important characteristics of geometric imagination. We believe that pupils who are engaged in mathematics by such specific activities have consequently better results in geometry. To confirm our hypothesis, we plan to continue in these activities, with larger pupil sample, we will observe more pupils and compare their results.

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INTERDISCIPLINARY RELATIONS OF SCHOOL MATHEMATICS AND BIOLOGY

IVETA KOHANOVÁ, IVANA ŠIŠKOVÁ

ABSTRACT. *In this article we discuss interdisciplinary relations of school mathematics and biology. We show examples of how to link mathematics to biology at both levels of secondary education be means of collection of 30 math tasks having biological context. Usage of these tasks was tested in four schools and the results show they have the potential to be an appropriate tool for developing cross-curricular teaching.*

KEY WORDS: *cross-curricular teaching, mathematics, biology, tasks*

CLASSIFICATION: *A93*

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Introduction

Nowadays it is an effort not only textbook authors but also politicians to develop pupils' competence regarding resolving problems from real life. The teaching of mathematics (but not only) should involve skills and functions that are part of everyday life. Students should be able to see things in context, think critically, argue and to use knowledge from several subjects. However, the reality is that knowledge of one subject most students are able to use only on this subject, and only to the time until the next lesson begins. Rarely it happens that pupils perceive how, when and which knowledge of physics, for example, can be used in a chemistry class. It is therefore important in teaching to promote cross-curricular teaching. The actual National Educational Programme [1] encourages teachers actively use interdisciplinary relations during the teaching and learning of pupils find links not only theoretical knowledge with practice, but also to other scientific areas.

In this article we will discuss interdisciplinary relations of school mathematics and biology. We show examples of how to link mathematics to biology at both levels of secondary education be means of collection of 30 math tasks having biological context. In verifying the benefits of an interdisciplinary approach in teaching, we tried to determine whether students are able to in addition to learning one subject also learn something from another object or whether they are able to link the knowledge acquired on different subjects in solving complex tasks. The results of this research should be the basis for teachers to find good practices that ensure maximum development of knowledge and skills of their students.

Interdisciplinary approach, cross-curricular teaching

Interdisciplinary approach is in pedagogical vocabulary defined as: „didactical approach in the teaching field to encourage cross-curricular teaching; solving special tasks that push students to integrate knowledge from different subjects; group work; creating of so-called integrated subjects or textbooks and other.“ [2] Authors of mentioned vocabulary understand cross-curricular relationships as "mutual respect between objects, understanding of the causes and relationships exceeding a subject framework and as medium for curricular integration".

Authors of [3] distinguish between even two possible modes of interaction between knowledge. They state that even the scientific knowledge of the individual sciences, there are certain relationships that can be divided into two categories: "... interrelationships - relationships between individual knowledge of various science disciplines, those are also called the term interdisciplinary relationships, and intrarelationships - relationships between individual knowledge the same disciplines of science, those are called the term intradisciplinary relationships." They call these types of interactions by term cross-science relationships. Whereas the content of subjects taught in schools reflects to some extent the hierarchy of scientific knowledge relevant field of science, it is clear that such interaction must exist between knowledge of individual subjects. "Analogous to cross-science relationships exist between school subjects educational ties that the term cross-curricular. These include cross ties - ties between elements of different systems of teaching subjects; often termed horizontal, and intra-subject ties - ties between the elements of the didactic teaching the same subject, often termed vertical." [3]

For use intra-subject links we could consider a hierarchical and spiral arrangement of curriculum established by National Educational Programme [4]. The pupil does not know; for example, simplify rational expression, if not familiar enough with fractions and expressions. Would not also make it to solve word problems where using linear inequations, if not based on the solution of linear equations, and the like. If a student does not know the composition of blood and blood characteristics of individual groups, then cannot understand the principle of blood donation. If a student does not know the characteristic of vegetation zones, then will not be able to list examples of plants and animals that inhabit it. As an example of the cross-curricular ties may be mentioned genetics. In order to determine the intensity of the reference character, which will be transferred from parent to offspring generation must student besides knowledge of the method of inheritance of the character also know at least the basic rules of combinatorics. To be able to determine the likelihood of occurrence of hereditary diseases in the offspring, he/she has to know the genotype causing disease, but must also know method of calculating the probability of the phenomenon. Without understanding the physical and chemical processes that accompany changes in energy is almost impossible to understand the process of photosynthesis and respiration. Cross-curricular ties are therefore just as important as those intra-subject ones.

Cross-curricular teaching of mathematics and biology

Mathematics and biology, both subjects are generally difficult for students. One cannot just learn mathematics without certain mathematical and logical thinking; on the other hand biology demands to know too many terms, characteristics, divisions, etc. The combination of these two subjects may therefore be even more challenging, not only for students, but also for teachers. To mathematics teacher who don't teach biology, facilitate their preparation for the lessons in which we promote cross-curricular teaching alluded to these two subjects, we have prepared a collection of 30 tasks having biological context. Given the limited scope of this article here are just a few of them, the first two for lower secondary education (A, B), and higher for the other two (C, D).

A) Human skin has several important functions. The most important are defensive capability against ingress of contaminants and mechanical resistance to environmental influences. The skin is also an important heat insulator of the human body. The researchers found that at 2.5 cm of human skin are 19 million skin cells. Calculate how many skin cells are located on the palm of your hand.

B) Blood in the human body is transported by blood vessel. There are 3 types of blood vessels - the arteries, which carry the blood away from the heart; the capillaries, which enable the actual exchange of gas between the blood and the tissues; and the veins, which carry blood from the capillaries back toward the heart. After expansion of all blood vessels in the human body, these occupy a total length of 150,000 km. How many times we could wrap them around the equator?

C) "The contrast agent, which is used in normal radiograph (intravenous pyelogram), is lacking in the milk of nursing mothers, and so the mother does not have to stop breast-feeding. However, if necessary, to nursing mother underwent a special screening of the lungs or bones it is necessary to use radioactive substance called technetium. It is a substance which has a short, only 6 hour half-life (the time in which half the dose leaves the body). If the concentration of the substance is less than 1%, so it is harmless for the child." Using information provided above, calculate for how many days after the lung screening with technetium mother can continue breastfeeding her child without compromising baby?

D) Snoring is an annoying problem that arises due to restriction of the airway and can lead to so-called sleep apnea, which is involuntary cessation of breathing during sleep. Research shows that one in eight boys snores and one in ten grinds teeth. What is the probability that among your peers is a boy who snores and also grinds teeth?

In verifying the benefits of a cross-curricular teaching, we tried to determine whether students are able to in addition to learning one subject also learn something from another object or whether they are able to link the knowledge acquired on different subjects in solving complex tasks. For this purpose we tested mentioned collection of tasks in the period September 2013 - March 2014 in four schools in Bratislava. We were not able to test all tasks, only 12 of them, but those for several times. Besides solving tasks with students in school, we have also created for each task control question. Control questions have nature of test items. In each question, students had to choose from four options, each of which has only one correct. Students may also choose an answer *I don't know* if they were not sure of the answer (thus we wanted to avoid betting). It is important to note that biological knowledge or attractions that are found in mathematical tasks, we tried to incorporate into tasks so that students submitted before them to learn in biology, so we tried to eliminate skew the results of our research because of previously learned knowledge. In order to find out whether students have really learned knowledge related to biology even when solving mathematical tasks, the testing by control questions held twice - once before students solved task in school and the second time with a certain time lag (at least four weeks) after solving this task. Here are examples of control questions related to mentioned tasks A, B, C, D.

A) Which of the following functions **do not ensure** human skin?

a) thermal insulation
b) mechanical resistance
c) defensive function
d) receiving oxygen
e) I don't know

B) Mark the claim that is **<u>incorrect</u>**.

a) Arteries carry the blood away from the heart.
b) Capillaries enable the exchange of gas.
c) Arteries carry the blood to the heart.
d) Veins carry the blood to the heart.
e) I don't know.

C) The time required for the left half of the dose of the body is called:

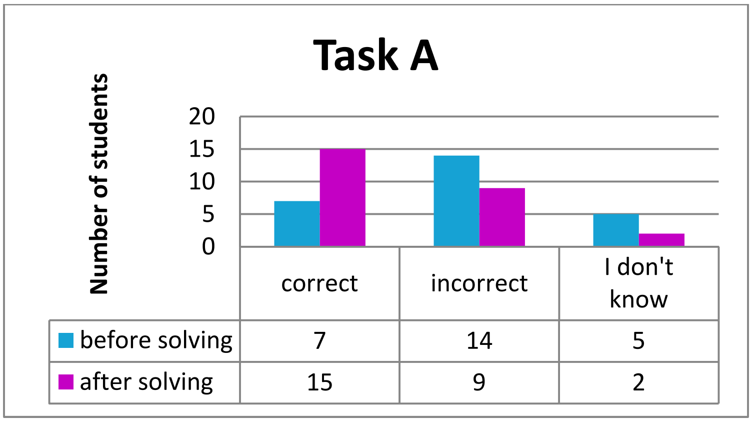
a) time of half-dose
b) time of half distribution
c) drug half
d) half-life
e) I don't know

D) What causes snoring?

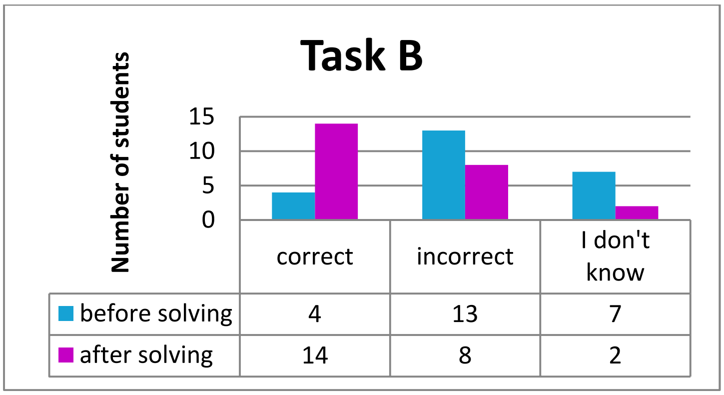
a) disorder in swallowing
. b) restriction of the airway
c) collapsed tongue in throat
d) pinched glottis
e) I don't know

Results and conclusion

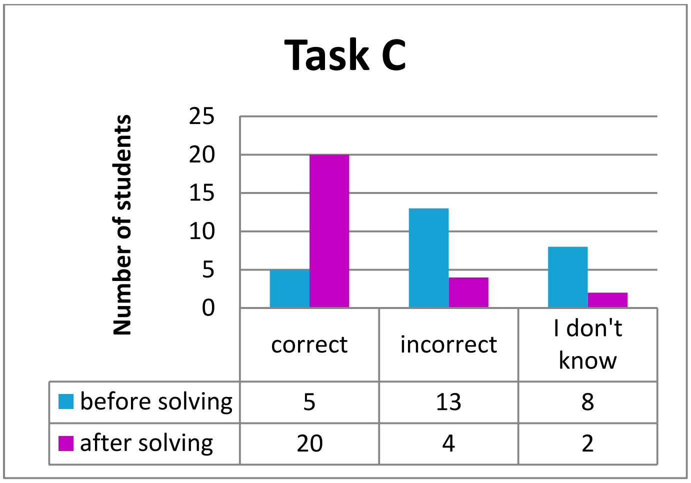
In following graphs and tables one could see results of testing for each of task A, B, C and D.



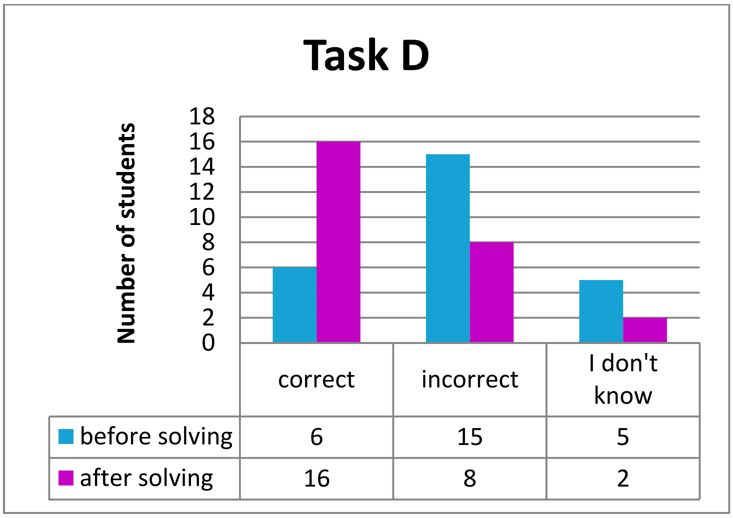
Task A	before solving	after solving	number of students
	correct	correct	5 (19,2 %)
	incorrect / I don't know	correct	**10** (38,5 %)
	correct	incorrect / I don't know	**2** (7,7 %)
	incorrect / I don't know	incorrect / I don't know	9 (34,6 %)



	before solving	**after solving**	**number of students**
Task B	correct	correct	3 (12,5 %)
	incorrect / I don't know	correct	**11** (45,8 %)
	correct	incorrect / I don't know	**1** (4,2 %)
	incorrect / I don't know	incorrect / I don't know	9 (37,5 %)



	before solving	**after solving**	**number of students**
Task C	correct	correct	4 (15,4 %)
	incorrect / I don't know	correct	**16** (61,5 %)
	correct	incorrect / I don't know	**1** (3,8 %)
	incorrect / I don't know	incorrect / I don't know	5 (19,2 %)



	before solving	**after solving**	**number of students**
Task D	correct	correct	5 (19,2 %)
	incorrect / I don't know	correct	**11** (42,3 %)
	correct	incorrect / I don't know	**1** (3,8 %)
	incorrect / I don't know	incorrect / I don't know	9 (34,6 %)

Results of testing tasks show that the majority of students at each of the selected tasks learned (memorize) the knowledge related to biology. Their responses when tested after their task' solving are mostly improved, respectively, are not worse. Based on the results of testing all 12 tasks can be assumed that assigning such tasks to the teaching of mathematics really effectively develop and promote interdisciplinary relations of school mathematics and biology. In further research it would be interesting to monitor whether students have mentioned biological knowledge in a time when it is taught in biology and how this affects their attitudes towards biology and mathematics.

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FACULTY OF NATURAL SCIENCES
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ACTA MATHEMATICA 17

THE ROLE OF DIGITAL MATERIALS IN DEVELOPING THE ESTIMATION ABILITY IN ELEMENTARY AND SECONDARY SCHOOL MATHEMATICS

LILLA KOREŇOVÁ

ABSTRACT. *One of the main tasks of school mathematics is to create the connection between school mathematics and the real world by viewing different real situations from a mathematical perspective and creating mathematical models. In real life we rarely need a precise calculation, most of the time an estimation is enough. Linking school mathematics to real life situations is one of the reasons of teaching students to estimate in mathematics. School mathematic activities, leading to estimates are an important part of the mathematic education. Digital environments, in particular the e-tests, can significantly increase the ability of estimation in students. During a research we have found out that the teachers have to achieve their goal with few available materials and their lack of time causes that mathematical tasks for estimation are mostly ignored. In this paper, we will point out the possibilities lying in increasing the student´s ability of estimation by digital technologies.*

KEY WORDS: *estimation, e-test, digital material, GeoGebra, HotPotatoes*

CLASSIFICATION: *D40, U50, U70*

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Introduction

One of the main tasks of school mathematics is to create the connection between school mathematics and the real world by viewing different real situations from a mathematical perspective and creating mathematical models. In real life we rarely need a precise calculation, most of the time an estimation is enough. Linking school mathematics to real life situations is one of the reasons for teaching students to estimate in mathematics. The State Education program for "Mathematics and the work with information"(ISCED2 and ISCED3) contains many references about "estimation". In the educational standard of the aforementioned document is stated that students should be able to estimate the length, surface and volume of solids, estimate the size of an angle, characterize something based on an estimation(even in circular diagrams),estimate the result of mathematical operations and estimate the result based on graphical presentation(tables, diagrams)

"With critical thinking, constructivist approach and experience a student should gradually build an estimation of a given mathematical problem" (Žilková, 2009)

During a research we have found out that the teachers have to achieve their goal with few available materials and their lack of time causes that mathematical tasks for estimation are mostly ignored. Nowadays students live in a digital world and digital technologies are getting more common even in schools therefore e-tests should be designed to develop the competencies of estimation in the students.

In a broad sense e-tests are electronic interactive materials, based on questions and finding answers, designed to measure not only knowledge, but to be suitable for teaching purposes too. E-tests can not only be used in classrooms equipped with computers and laptops but they are getting more commonly used on tablets too. One of the best known e-test making software which is available for free is HotPotatoes.

Fulier J. (Fulier, 2005), Partová E. (Partová, 2011) and Žilková K. (Žilková, 2009) deal with the topic of the application of digital technologies in education.

Types of estimation tasks

Tasks in the teaching of mathematics designed to develop the students competence in estimation are divided into several groups.

Concerning the purpose we can divide tasks to obtaining a particular knowledge (estimations of extent, count) and to tasks where using logical thinking the student estimates correctly.

Estimation tasks according to Samková could be divided to:

- Numerosity estimates, which can be divided also to:
    - 1-dimensional (estimation of the number of beads on a string)
    - 2-dimensional (estimation of the number of people in a square, the number of cars in the parking lot)
    - 3-dimensional (the estimated number of candies in a jar, the number of bricks on a pallet)
- computational estimates
- measurement estimates which can be also divided to:
    - 1-dimensional (estimation of length, distance, height, size, time)
    - 2-dimensional (estimation of content, surface, angle size)
    - 3-dimensional (estimation of volume, mass) (Samková, 2013)

In terms of integration tasks into themes we can divide estimation tasks to:

- Numbers and variables
- Relations, functions, tables, charts,
- Geometry and measurements
- Combinatorics, probability, statistics.

According to Eszter Herendiné-Kónya: "The estimation capabilities are not only essential in our daily lives but are also a great help in solving mathematical problems, because knowing the approximate value makes the verification process easier." (Herendiné-Kónya, 2013)

The four dimensions of understanding the concept of area measurement:

1. Skill-algorithm understanding is choosing an appropriate algorithm to calculate the area depending on the plane figure and the given sizes.
2. Property-proof understanding includes derivations of the basic formulas for the areas of triangles and other polygons, relations between area and perimeter of the same figure etc.
3. Use-application understanding includes area measurement in everyday life, applications in complex problems etc.
4. Representation-metaphor understanding includes area measurement with congruent tiles, cutting and rearranging polygons, area representation with an array of dots etc. (Herendiné-Kónya, 2012)

Levine D. (Levine, 1982), Krajčiová J. (Krajčiová, 2005) and Herendiné-Kónya E. (Herendiné-Kónya, 2012) are dedicated to the issue of the estimation skills in school mathematics.

In the following part we will show some e-material examples:

<u>The estimation of a number (quantitative estimation):</u>

In this type of tasks the students have to estimate the number of the objects under the given time. Objects (dots, stars, beads etc.) could be sorted or scattered around. This type of tasks can be found on the Internet (Figure 1), but these are rarely free.
The students have to estimate the number of objects under the given time.

Figure 1

Such tasks can be made in the software HotPotatoes (Figure 2).In this task, the students have to estimate the number of cars in the parking lot. The test contains several different questions sorted by their increasing difficulty. The increasing difficulty is achieved not only by increasing the amount of cars but also by their uneven layout in the parking lot.
This correct answer is decided by this interval,

$$\left(s-\frac{s-o}{s};s+\frac{s-o}{s}\right),$$

where s is the correct number and o is the estimation of the student.
Feedback is important for the students, which includes not only the correct number of the cars but also the extent of their accuracy, for example: "The difference between your estimation and the correct answer is less than 10! Excellent!"

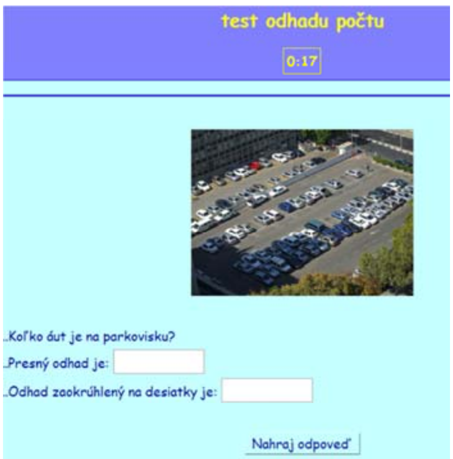

Figure 2

E-materials made for estimating numbers can be created using GeoGebra.

Interesting presentations were made by Samková (The application of GeoGebra in estimation tasks, 2013)

Estimation of extent:

There are several tasks on the Internet, where students have to estimate the relative length of a segment (Figure 3) and the angle in degrees (Figure 4)

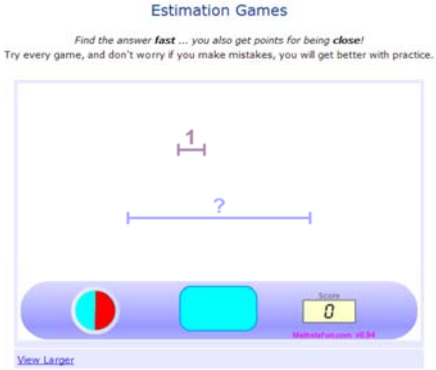

Figure 3

Figure 4

On the website "Planéta vedomosti", purchased by the Slovak Ministry of Education and which is also accessible to all schools for free, contains several interesting puzzles.
One of these is the estimation of metric units of lenght. (Figure 5)

Figure 5

<u>Computational estimates:</u>

The most common tasks for calculations are the estimations of the results of mathematical operations with rounding. Its a very important skill that needs to be practised by the students during every types of mathematical operations. These tasks include, for example estimates for:

- addition, subtraction, multiplication and the division of natural numbers (Figure 6)

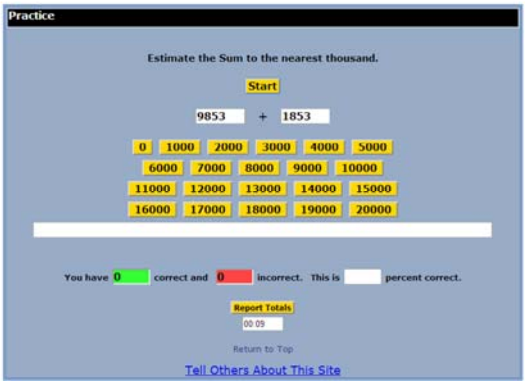

Figure 6

- addition, subtraction, multiplication and the division of decimals
- addition, subtraction, multiplication and the division of integers
- the calculation of square roots.

Interesting tasks are also where the students have to estimate the approximate location of numbers on the number line. The students must round the number and then decide to which number marked on the number line is their number closer. During these tasks they must solve inequalities. Such tasks can be found on the Internet (Figure 7) too, or can be created by HotPotatoes or GeoGebra.

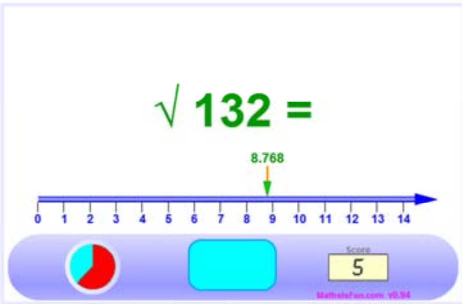

Figure 7

While creating these materials its important to keep in mind that that the feedback should assist the student in finding the right strategy for the estimation.

Teachers opinion poll

In February 2014 we proceeded with an opinion poll. Our goal was to determine the current situation of using estimation tasks in teaching on elementary and secondary schools by theme, type of the school and the used books. We were also interested in the possibility of using digital technologies during the lessons of mathematics. The poll was focused on gathering the teachers opinions whether estimation tasks are important and what kinds of tasks do they give to their students. The poll was realized via an electronic questionnaire. 130 teachers participated (108 women and 22 men), mostly absolvents of continuous teacher education.

We present a few interesting findings:

On the question „According to your opinion, are estimation tasks important in elementary and secondary school mathematics? " 48% answered with yes and 42% with maybe yes – that means 90% of the participants answered positively. (Figure 8)

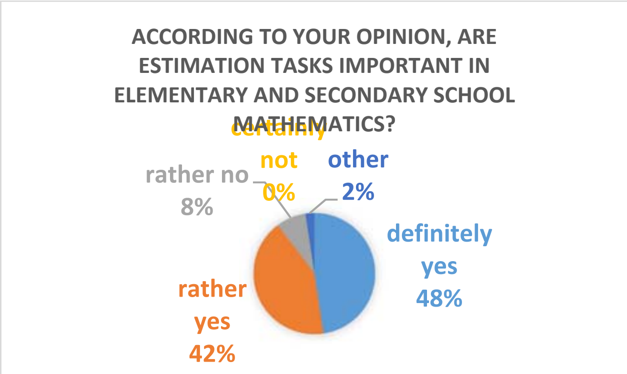

Figure 8

On the question "Is there a sufficient amount of estimation tasks in the textbooks?" 53% answered by no and 22% with definitely no – that means 75% negative.

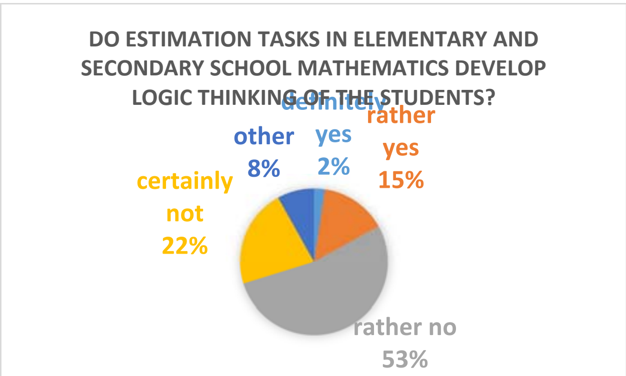

Figure 9

On the question "Do estimation tasks in elementary and secondary school mathematics develop logical thinking of the students?" 56% answered with definitely yes and 35% yes – that means 91% answered positively.

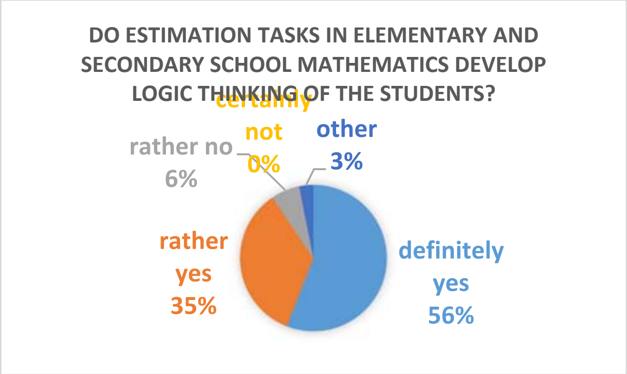

Figure 10

With this poll we have proven our hypothesis, that teachers consider using estimation tasks as important and feel the absence of these types of tasks from the textbooks.

Conclusion

With computing technologies on a rise, people put less emphasis on quick and accurate mechanical calculations, rather more importance is given to the students ability to understand each step of the calculation and algorithm. Therefore, tasks where students must reckon in their head and estimate are getting more popular. Tasks which develop the estimation competence of the students therefore are an important part of school mathematics and e-tests are a great way to increase their estimation skills even further. Students can solve these tasks individually, at their own pace and get immediate feedback. It´s important where the students can make an "estimation strategy" for the tasks to get immediate feedback in the form of instructions for an appropriate strategy. Such e-tests can be found on the internet mostly in English. A great software to make e-tests like this is HotPotatoes.

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STATISTICAL PROCESSING OF THE POSTTEST RESULTS OF 8TH GRADE OF PRIMARY SCHOOL OF KEGA 015 UKF – 4/2012 PROJECT

MÁRIA KÓŠOVÁ [1], MARTA VRÁBELOVÁ

ABSTRACT. *The paper deals with statistical processing of posttest results, which was written as part of project KEGA 015 UKF – 4/2012. The aim of this project was to compare knowledge level of pupils from 8th grade of primary school in the treatment and control group in this school year 2012/2013. By comparing these groups we want to verify hypotheses as are comparing the knowledge of pupils from treatment school with Hungarian and Slovak teaching language, the pupils with and without failure learning and the level of boy knowledge and the level of girl knowledge. This paper deals with the reliability of the posttest and the item analysis of the test also.*

KEY WORDS: *KEGA_015_UKF_–_4/2012, statistical processing of the posttest results, tasks of real-life context*

CLASSIFICATION: *B12*

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Introduction

This paper offers information about statistical processing of posttest results, which was written within the solution of project KEGA 015 UKF – 4/2012 named *Increasing key mathematical competencies II – Alternative learning programs in mathematics for primary schools in line with the aims of the new national education program and in line with increasing mathematical literacy under the impact of PISA.* Aim of the project in the school year 2012/2013 was creating new learning materials in teaching mathematics and verification of effectiveness of teaching using these materials in the 8th grade of primary school. This project follows project KEGA 3/7001/09 with the same name, which was dedicated to creating learning materials and verification of their effectiveness in the 5th and 6th grade of primary school. These materials are aimed at increasing key mathematical competencies by solving tasks inspired by real – life problems and so are aimed also at preparing pupils for international testing.

An experiment is being realized within the project. This experiment started with the random partition of schools involved in research into treatment and control group and then with writing the pretest. Results of pretest can be found in the paper [7]. Learning materials inspired by real – life problems are for treatment schools prepared in every year of project solution. Content of materials includes every of five education areas as they are defined in the National education program. Posttest is written in May every school year. Statistical processing of posttest results in previous school years are available in paper [8], [4] and [5].

[1] Corresponding author

Main research hypothesis and research sample

Main research hypothesis is:

Prepared materials effectively contributed to increase key mathematical competencies of pupils 8^{th} grade of primary school.

Research sample consisted of 544 pupils of 8^{th} grade of primary schools of four Nitra region districts. Some of these schools are schools with Hungarian teaching language.

Methodology and research tools

We used experiment as a research method. Schools were grouped in the treatment and control group randomly. In the school year 2012/2013 treatment group consisted of 11 schools and control group of 12 schools. We used didactic tests as a research tools – pretest and posttest. Pretest was written only at the beginning of the experiment (it means in the 5th grade of primary school). Posttest is written in every year of experiment duration.

Posttest and additional research hypothesis

Posttest for 8^{th} grade of primary school contained 6 tasks. Every task consisted of two subtasks. All questions in tasks were open. Content validity of the test was assessed by teachers of 8^{th} grade of primary schools. At first the test was tested in one school and according to its results some of the tasks were modified. Pupil could get maximum of 5 points for each of subtasks, total maximum of 30 points (total). Task named "*Age of dog breeds"* (author PaedDr. K. Cafiková) was focused on logic. Task named *"Consumption of a car"* (author doc. RNDr. P. Vrábel, CSc.) was focused on area of expression. Task named *"Game Drawing of number"* (author PaedDr. Eva Uhrinová) was focus on probability. Task named *"Traffic roundabout"* (author Mgr. Zuzana Vitézová) was focused on geometry. Task named *"Candles"* (author Mgr. L. Lednický) was focused on area of functions. And task named *"Media in our school"* (author Mgr. Mária Kóšová) was focused on statistics. Due to the scope of the article we cannot provide the whole test, which was used. The test is available in www.fss.kega.ukf.sk.

The following hypothesis is verified according to posttest results:

Hypothesis 1: *Level of knowledge of pupils in the treatment group is significantly higher than the level of pupils in the control group.*

In addition to this hypothesis, we set out to verify also these hypotheses:

Hypothesis 2: *The level of knowledge of pupils in treatment schools with Slovak teaching language is not significantly different from the level of pupils in treatment schools with Hungarian teaching language.*

Hypothesis 3: *The level of knowledge of boys in the treatment schools is not significantly different from the level of knowledge of girls in the treatment schools.*

Hypothesis 4: *The level of knowledge of pupils with learning disorders from treatment schools is significantly different from the level of other pupils from treatment schools.*

The results of the posttest

Due to the hypotheses we compare the average test scores in these groups : treatment and control (E and K) , treatment schools with Slovak teaching language (ESJ) and with Hungarian teaching language (EMJ), group of girls from treatment schools (EZ) and boys from treatment schools (EM), group of pupils without learning disorders from treatment schools and a group of students with learning disorders from treatment schools (ENo and

EYes). Descriptive statistics of the compared groups are summarized in table 1.

group 1, group2	Descriptive statistics – average test score, (average), standard deviation (st. dev.), count of valid cases (count)					
	average group 1	average group 2	st. dev. group 1	st. dev. group 2	count group 1	count group 2
K, E	9,96	17,93	7,35	7,54	323	221
EMJ, ESJ	18,73	16,49	8,25	5,82	142	79
EZ, EM	18,77	17,07	7,65	7,35	111	110
ENo, EYes	18,23	6,5	7,93	6,45	13	12

Table 1: Descriptive statistics of posttest of 8th grade of primary school in the project

Since the *p*-value of normality test (Shapiro - Wilk test, Kolmogorov-Smirnov test) are for groups E, K, EMJ, EZ, EM less than 0.05, scores in these groups can not be considered to be normally distributed. Scores in groups ESJ, EYes, ENo can be considered to be normally distributed.

In view of the above, we verify Hypothesis 1, Hypothesis 2 and Hypothesis 3 using the non-parametric Mann-Whitney U test and Hypothesis 4 using the t-test.

gr.1, gr.2	Sum of rank group 1	Sum of rank group 2	U	Z	p	Z adj	p	Count group 1	Count group. 2
K, E	68534	79706	16208	10,82	0,00	10,82	0,00	323	221
EMJ, ESJ	17085	7446	4286	-2,90	0,004	-2,908	0,004	142	79
EZ, EM	13212,5	11318,5	5213,5	1,88	0,06	1,88	0,06	111	110

Table 2: Results of Mann – Whitney U test

gr.1, gr.2	average group1	average group 2	t	df	p	F	p	st. dev. group 1	st. dev. group 2
EYes, ENo	6,5	18,23	4,04	23	0,0005	1,51	0,5	6,45	7,93

Table 3: Results of t-test

A graphical representation of the average test score together with 95% confidence intervals for each group can be seen in Figures 1a - 1d. For comparison, Figure 1b, 1c and 1d we present and illustrate also test scores of the control schools.

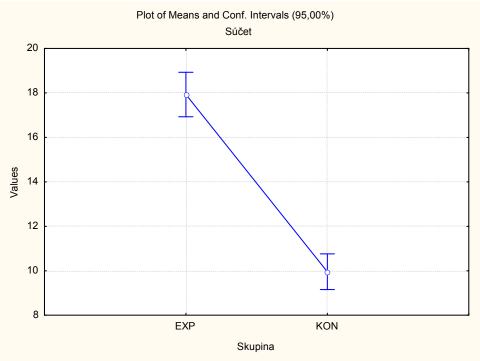

Figure 1a: Graph of average for E and K

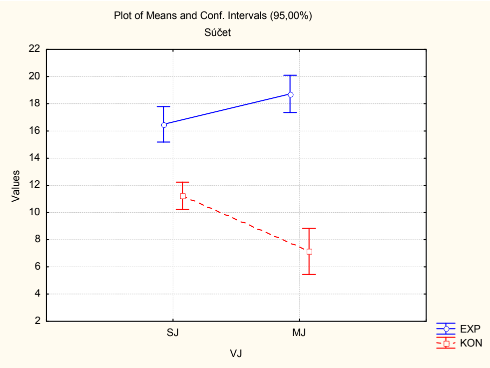

Figure 1b: Graph of average for SJ and MJ

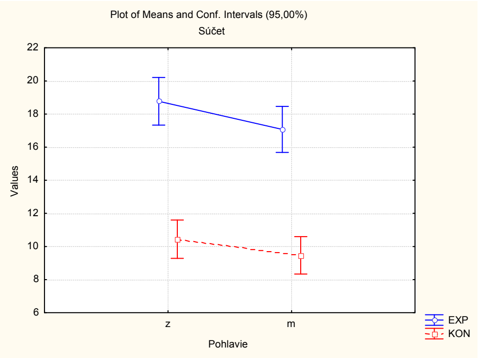

Figure 1c: Graph of average for Z and M

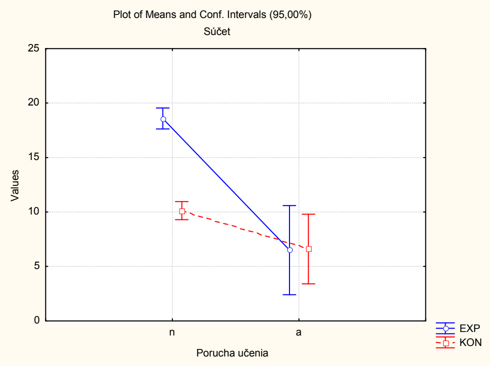

Figure 1d: Graph of average for No (n) and Yes (a)

According to results of M-W U test (Z adj. = 10.83, p = 0.000) we reject statistical hypothesis Mean value of test score is the same in the group E and K, and also according to graphical representation of average posttest scores (Figure 1a) it is clear that the average posttest score of the treatment group (17,93) is statistically higher than of the control group (9.96). Therefore the Hypothesis 1 is valid.

Results of *M-W U test* show that there is significant higher posttest score in the treatment group of school with Hungarian teaching language (18,73) than in the treatment group of school with Slovak teaching language (16,49) as well. If we look at Figure 1b, we can realize that there is probably significant higher posttest score in the control group of schools with Slovak teaching language than in the control group of schools with the Hungarian teaching language. In light of this, we divided data file into the four groups (ESJ, KSJ, EMJ, KMJ). We compared posttest scores in these groups using the multiple comparisons. We used Kruskal–Wallis one-way analysis of variance (H=123,8139 a p-value=0,000) and post-hoc test to Kruskal–Wallis test. We found that differences between almost all of groups are significant except difference between groups EMJ and ESJ. The p-values are in the Table 4. This conclusion is a little bit different from previous one. But, we are focused on comparison groups EMJ and ESJ especially, so we accept previous results.

Therefore, we can say that Hypothesis 2 is valid. It means that level of knowledge of pupils from treatment group of schools with Hungarian teaching language is significantly higher than level of pupils form treatment group of schools with Slovak teaching language. Very interesting is also that level of knowledge of pupils from control schools with Slovak teaching language is significantly higher than level of knowledge of pupils from control schools with Hungarian teaching language (*p*-value = 0,000944). Therefore we can say that in schools with Hungarian teaching language became to higher improvement.

	KMJ (7,14)	EMJ (18,73)	KSJ (11,23)	ESJ (16,49)
KMJ (7,14)		0,000000	0,000944	0,000000
EMJ (18,73)	0,000000		0,000000	0,621231
KSJ (11,23)	0,000944	0,000000		0,000009
ESJ (16,49)	0,000000	0,621231	0,000009	

Table 4: Nonparametric multiple comparisons

The results of the *M- W U test* (Z adj. = 1.88, p = 0.06) also shows that difference between average posttest scores in the group of boys from the treatment schools (17,07) and group of girls from treatment schools (18,77) is not significant. Hypothesis 3 is valid.

We made a random selection of 13 students from group ENo in order not to compare data files (EYes and ENo) with very different range. The result of *t-test* for comparison of these two groups shows that there is a significant difference in the level of knowledge among groups EYes and ENo. Hypothesis 4 is valid.

We placed descriptive statistics of posttest score in the table 5 and item analysis (average, modus, median, standard deviation, % of pupils, which obtained maximum score from task, % of pupils, which obtained score 0 from task) in the table 6. We can see that in every task the rate of pupils with maximum score from task was lower than 80%; it means that none of the tasks is "suspect". Modus is equal to maximal score (score 5) in the task named "*Age of dog breeds"* and *"Traffic roundabout"*.

Variable	N	average	median	modus	count of modus	min	max	st. dev.
Sum	544	13,19	13	5	32	0	30	8,39

Table 5: Descriptive statistics

	Descriptive statistics								
	average	med	mod	count mod.	min	max	st. dev.	% pupils min	% pupils max
Age of dog breeds	2,69	2	5	185	0	5	1,96	22,61%	34,01%
Consumption of a car	1,64	1	1	231	0	5	1,40	17,46%	7,72%
Game Drawing of number	1,53	1	0	269	0	5	1,87	49,45%	13,42%
Traffic roundabout	2,64	3	5	151	0	5	1,95	25,18%	27,76%
Candles	2,17	2	0	228	0	5	2,14	41,91%	28,68%
Media in our school	2,52	2	2	222	0	5	1,68	15,81%	23,35%

Table 6: Item analysis of posttest

Pupils achieved the highest average success rate of right solutions on solving the task *"Age of dog breeds"*. Conversely, the task named *"Game Drawing of number"* was the most difficult for pupils.

We computed reliability coefficient (Cronbach's alpha = 0,85) in order to check the degree of reliability of posttest. We should also compute value of Cronbach's alpha after deleting one of tasks. If value of Cronbach's alpha becomes higher, this task makes reliability of test lower. Values of Cronbach's alpha after deleting one of task are shown in table 7. We can say that posttest was reliable enough.

	Age of dog breeds	Consumption of a car	Game Drawing of number	Traffic roundabout	Candles	Media in our school
alpha after deleting	0,82341	0,83542	0,82421	0,81956	0,81826	0,83085

Table 7: Value of reliability coefficient after deleting one of tasks

Conclusion

We have shown that posttest which was made by us for pupils of 8th grade of primary school was reliable adequately.

According to its results it can be concluded that materials prepared for teachers and their pupils of 8th grade of primary school helped to increase their key mathematical competences effectively. It turned out pupils from the treatment schools with Hungarian teaching language achieved significantly better results than pupils from the treatment schools with Slovak teaching language. In contrast, pupils from the control schools with Slovak teaching language achieved better results than pupils from control schools with Hungarian teaching language. It can be assumed that this may be due to the fact that in schools with Hungarian teaching language teachers worked with prepared materials more than in schools with Slovak teaching language. Difference between levels of knowledge was not significant with regard to the gender of pupils in the treatment schools. The expected result about significant higher level of knowledge of pupils from treatment school without learning disorders occurred.

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THE DEVELOPMENT OF COMBINATORIAL SKILLS OF THE LOWER PRIMARY SCHOOL PUPILS THROUGH ORGANIZING THE SETS OF ELEMENTS

RADEK KRPEC

ABSTRACT. *The article is aimed at studying the organizational principles through the use of the preschool pupil and the lower primary school pupils within the scope of one experiment, and also at the forthcoming experiment with the upper primary school pupils. The experiment with the lower primary school pupils deals with their ability to spread a set of cards in such a way that they are able to memorize the layout. Within the scope of the experiment we observed not only the effectiveness and principles of the organization of cards, but also the originality of the layout. The other part of the article deals with the forthcoming experiment with the upper primary school pupils regarding combinatorial tasks solving.*

KEY WORDS: *organizational principles, combinatorics, organization of elements.*

CLASSIFICATION: *B10*

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1. Introduction

Since 2012, one of the areas of our research is finding strategies for solving tasks related to combinatorics in lower and upper primary school. "Combinatorial thinking is based on the ability to organize the elements of the set into transparent charts, graphs, diagrams, and lists" ([3]). In pupils of lower primary school the development of the ability to organize set of elements is a propaedeutic to combinatorics. Those abilities are necessary to be developed on the lower primary education level ([1], [5]). This propaedeutic is properly worked out in the textbooks of mathematics for lower primary school by professor Hejný [2]. As far as the lower primary school pupils are concerned, during the experiment we observe their ability to organize sets of elements or sets of phenomena. Despite the fact that the teachers try to explain combinatorial thinking to the upper primary school pupils and the secondary school pupils, the teaching process tends to follow the pattern of solving tasks according to a particular algorithm: pupils choose the right formula, solve the tasks and deliver the answer. Therefore it is important to develop abilities to solve combinatorial tasks in the primary school [4]. The aim of our ongoing experiment with the upper primary school pupils is to observe solving strategies and the changes of the strategies for solving combinatorial tasks.

2. The Experiment with Lower Primary School Pupils

The experiment, which included one pupil in preschool age named Adélka, and two pupils of first year named Markéta and Marek, two pupils of second year named Klára and David, and one pupil from third year named Adam of the lower primary school, took place in March 2013. The pupils took part in the experiment individually. The goal of this experiment was to find out in which way pupils organize objects. During the experiment we observed some of the didactic parameters such as environment, used language, and mainly the organizational principle.

2.1 The Description of the Experiment

The experimentalist and one pupil take part in the experiment. The experimentalist has four sets of cards, each of them in two identical versions.
Set A. Six different animals: rooster, hen with chickens, monkey, bear, dog, hare.
Set B. Yellow cards with numbers: 1, 2, … 11, 12 (See Figure 1).
Set C. {blue, yellow} × {all six sides of a die} (See Figure 2).
Set D. {blue, yellow, green} × {bear, monkey, rooster, dog}.

Figure 1: Set B


Figure 2: Set C

The game is optional, pupils can end it anytime and go to play.

2.2 The Schedule of the Experiment

Action 01. Introductory interview. The experimentalist gives a pupil the set A and says: “Here you have a set of six cards. Spread the cards on the table. Then, we will turn the cards upside down and you get a second set of cards, which you will be supposed to order in the same way”. The experimentalist gives the pupil the set A. In case the pupil says or asks anything, the experimentalist reacts accordingly. The pupil spreads the cards. A few seconds after the last card is laid on the table, the experimentalist asks: “Do you remember the sequence of the cards? Can we turn them upside down?” The experimentalist waits for pupil’s approval and turns the cards. If the pupil joins him/her in the process, the experimentalist thanks them.

Ex: “Here you have the second set (gives the pupil the second set). Spread it in the same way as the first one”.

The pupil spreads the second set of cards, the experimentalist remains silent. After the pupil finishes spreading the cards, the experimentalist asks him/her: “Do you think this is correct? Shall we start turning the cards, or do you want to change anything?” Ex: “Let’s turn the cards upside down.” The experimentalist turns the second set of cards upside down. The experimentalist evaluates the pupil’s result. If the result is successful (which is expected), the emotional part of the evaluation is intensified.

Action 02. The experimentalist repeats the Action 01 experiment, but turns the cards upside down immediately after pupil finishes spreading the first set (about which he/she notifies the pupil beforehand).

Action 03. The experimentalist pulls out two sets B and explains to the pupil that they will play the same game as with the set A. Moreover, if the pupil succeeds in spreading the two sets in the same way, he/she is given a point. If not, the experimentalist is given a point. The game and evaluation will take place. Other actions with the same set follow; now the pupil is expected to spread the cards in a different way. These actions are then repeated once or twice according to pupil’s needs and interest.

2.3 Didactic Parameters

During the experiment we observed several didactic parameters such as environment, used language, and mainly the organizational principle.

As far as an environment is concerned, we can choose from three options: a) semantic, b) structural, c) hybrid. As for the language, mainly the manipulative language was used (combined with the graphic language). Our main interest was the organizational principle. We looked for the organizational principle in all tasks, in fact for organizational principles, since several organizational principles might have occurred during a single action.

2.4 The Results of the Experiment

First we analyzed the set B experiment. We left out the training actions. Now we will gradually focus on experiments with pupils from the preschool age to the third year.

The preschool age pupil's name is Adélka. She spread the set in a creative way (she followed a pattern that enabled her to memorize the sequence). We can conclude that her layout is not influenced by working with numbers in lower primary school. She spread the set B as follows:

1	2	3	4	5
6	7	8	9	10
11	12			

The layout in the next action was again from 1 to 12, but in this case it was L-shaped.

The set C layout was no longer creative. The cards were spread in a way for her to remember the sequence. First the numbers from 1 to 6 in two lines, the first line being blue, the second line yellow. In the next game, she spread the blue cards into two lines, 1 3 5 in the first line, 2 4 6 in the second, and then she spread the yellow cards in the same way underneath the blue cards.

She spread the set D into the 4 x 3 pattern. She divided animals into lines and colours into columns.

The lower primary school pupils, Marek and Markéta from the first year, had different card spreading strategies. Marek chose the most space-saving layout. He spread the set B cards in ascending order or in descending order. Markéta, on the other hand, was more creative. She tried to spread the cards in a way to remember them correctly, but she also tried to add something that would make the layout more interesting or more demanding. As for the first try, she spread the cards as follows: 1 3 2 4 5 7 6 8 9 11 10 12. In this case, however, Markéta was not be able to spread the second line in the same manner, as the end of it may become disrupted by double figure numbers, and she spread the line as follows: 1 3 2 4 5 7 6 8 9 11 12 10. In the following game she chose a less demanding strategy (even if it is still not completely easy) and spread the cards as follows: 1 2 3 4 5 7 6 8 9 10 11 12. We do not know if the switch of numbers in the middle of the line was intentional, or if it was accidental and she only remembered which two numbers she switched. She spreads the second set in the same way. As far as the set C is concerned, Marek's economy of the card spreading and Markéta's creativity once again become apparent. Marek always puts cards of the same colour together and orders them either in an ascending or in descending order within the scope of one colour (e.g. 6 5 4 3 2 1 1 2 3 4 5 6). Markéta spreads the cards in an ascending order from the edge to the center; odd numbers from the left, even numbers from the right; yellow cards first then the blue ones: 1 3 5 1 3 5 6 4 2 6 4 2. She remembers this layout very well. Therefore, in the next game she chooses a more demanding strategy: 2 2 3 3 4 4 5 5 6 6 1 1. Even though the colours do not alternate regularly, Markéta is able to remember the sequence. In the first set of D game, Markéta once again aims for creativity, but is unable to remember the layout. Therefore, in the next game, she chooses an entirely economic strategy, similar to the one Marek chose in all set D games. The example of an economic layout:

Figure 3: Example of the economic strategy

The next ones who took part in the experiment were Klára and David – second year pupils. Once again, differences in the spreading strategies (organizational principles) are apparent. Klára chooses the most economic way of card spreading. As far as the set B is concerned, she chooses the sequence of 1 to 12 either in ascending or in descending order. David chooses much more demanding layout. In individual actions, he spreads the cards gradually and in pairs: e.g. 1 9 11 2 8 4 7 5 10 12 6 3. When the second spreading comes, the problem occurs with remembering the sequence of the pairs (e.g. 4 8) or with confusing the entire pairs. In the third game with this set, he chooses the layout that is easier to remember: 12 11 … 2 1. As for the set C, Klára once again chooses the economic layout: the cards of the same colour are put together; there is regular alternation of colours in a line with numbers being lined up either in ascending or in descending order. David once again chooses not so much economic layout, the result of which is that he is not able to remember it. The example of his layout is as follows: 1 1 6 6 5 5 4 4 2 2 3 3. A similar situation occurs in other set C games. It is not before the final game that David chooses the ascending sequence of cards with regular alternation of colours. As far as the set D is concerned, Klára once again chooses the easy organizational principle: she puts the same animals together; and chooses either alternations of colours in lines or the Cartesian layout 3 × 4. David too chooses the easy layout for this set.

The last one to take part in the experiment was Adam, a third year pupil. In the first set of B game, Adam uses the effective layout of cards. He spreads the cards into two lines: the first line having odd numbers in it, the second line having even numbers in it; numbers in both lines are lined up in ascending order. In the second game, however, Adam violates his organizational principles, the result of which is that he does not remember the first layout when doing the second one. Therefore, in the third game he chooses an entirely effective layout: puts all the numbers to one line and lines them up in ascending order from 1 to 12. In the first set of C game he once again uses a certain organizational principle for numbers, but uses no such principle for colours. This results in his not remembering the layout. The layout is as follows: 2 2 4 4 6 6 in the first line, and 1 1 3 3 5 5 in the second line with colours not repeating regularly. In the second game with this set Adam uses regular alternation of colours. The third set of C game is interesting. In this game Adam tries to use suitable organizational principles but at the same time tries to add creativity to the game. The layout is as follows: there is a line of in ascending order lined up yellow cards, which is interrupted by a column of blue cards lined up in descending order:

Figure 4: A line of in ascending order lined up yellow cards, which is interrupted by a column of blue cards lined up in descending order

In the set of D games Adam follows the organizational principles: the same animals in lines or columns, and the cards of the same colour in lines or columns. He uses either the 3 × 4 or 4 × 3 layouts.

2.5 The Summary of the Experiment with the Preschool Pupil and the Lower Primary Pupils

From the comparison of the experiments it is evident that the organizational skills used by pupils differ considerably in terms of structure, effectiveness, and difficulty. Some pupils (Marek, Klára) aim for the easiest (most effective) principle of card spreading; the one they can remember well. Other pupils (Markéta, David, Adélka, Adam) use the effective layout only after failures in the early games. Markéta chooses effective organizational principles into which she tries to add a creative aspect. This results in the violation of simple organizational principles. As far as David is concerned, we think that his failures are caused by his searching for suitable organizational principles, not by trying to make the game more interesting. Further we can notice, that while Markéta prefers to choose linear strategy in spreading the cards, Adélka and Marek with the set of C and D game choose mainly the strategy of Cartesian layout. At the beginning David keeps the linear layout but with the last set he chooses the Cartesian layout of cards, the same as Klára. The experiments prove that the pupils were able to find the organizational principles that helped them win the game every time. As the didactic parameters are concerned, from the used language mainly the manipulative in combination with graphic language occurred. The differences were mainly in the fact that Adélka, Markéta, Klára and David looked at the cards first and then they started to spread them on the table. On contrary Marek and Adam started to spread the cards immediately from hands, without looking at them first. That was one of the reasons why David had problems with finding a suitable organizational strategy. The environment was in this case hybrid, we worked on experience with the games of pairs type (semantic environment), the cards were adjusted for our experiment (structural environment).

3. The Experiment with Upper Primary School Pupils

Following the previous research, we will continue with upper primary school pupils. In this experiment we will deal with solving of combinatorial tasks. The research will have three phases. In the first phase, an experiment with individuals will be conducted. We will observe how their strategies change during solving of a combinatorial task, and how their solving strategies change with regard to solving the previous tasks. We will also be interested in which strategy of organizational principle they use to solve a combinatorial task. Besides, we will observe whether pupils are able to discover isomorphisms in solving of the individual tasks. In the next experiment, we will work with a group of pupils in a mathematical club. Apart from the aforementioned goals, we will observe the changes of strategies of the pupils who watch another pupil solving the task (e.g. at the blackboard); and the change of strategy for solving a combinatorial task in a group. We will want to apply some of the experiments conducted in a group of pupils to the entire class.

4. Conclusion

The aim of these experiments is to find out how to prepare the lower primary school pupils so as they are ready to solve tasks of combinatorial character on the upper primary school level. As far as the upper primary school pupils are concerned, we intend to focus on the best way of preparing them for combinatorial tasks. The basis of the education process is a teaching method aimed at scheme building, which has been our main research area for the past several years. The research has been conducted in cooperation with the Department of Mathematics and Mathematical Education of the Faculty of Education of Charles University.

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TESTING OF IRRATIONAL NUMBERS AT THE HIGH SCHOOL

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ABSTRACT. *This report is research on understanding of irrational numbers. Specifically, I focus here on how irrational numbers can be represented and how different representations influence participans´ responses with respect to irrationality.*

KEY WORDS: *rational and irrational number, pedagogical experiment, research, understanding.*

CLASSIFICATION: *B10*

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Introduction

Real numbers express quantitative properties of the real world. Therefore, the subject matter of real numbers included in mathematics textbooks for all types of schools. Studying at high school assumes mastering learning content in mathematics at a level that students could attend college. He should prove effective to acquire and process the information and mathematical form to real situations. He should understand the meaning and value of mathematics in the past and at present, get to know possibilities of application Mathematics in the Sciences and Humanities. Pupil relationship in high school being built through examples tackled.
The role of the teacher is to explain the given problem by a particular form to choose such tasks that attracted interest among students. I teach mathematics at university for 17 years. During my practice I found out that in explaining the subject matter is usually pupils meet with several sample solutions jobs. They then by copying these procedures solve tasks. Based on these observations I made the hypothesis, which I verified using the test.

Research setting

The hypothesis, I tried to verify by using the test. The main objective of these studies was to examine the knowledge of irrational numbers of high school students. I wanted to find out by testing as students distinguish rational from irrational numbers, operations with rational and irrational numbers, as well as understanding of the notion irrational number. For the purpose of experiment I randomly chose the group of 59 students from the 4-th grade of gymnasium from the Zilina region. The group was further divided into three groups. All the students were taught the same thematic unit.

Test

1. List the prime numbers from 1 to 30. Make of them irrational numbers.
2. Decide which of the following numbers are rational and the irrational:
   $\sqrt{0{,}25}$; $1{,}213\overline{213}$; π; $3{,}21875143\ldots$; $\sqrt{150}$; $\sqrt{900}$; $\sqrt{0.28}$; $\sqrt{25}$; $7{,}3284317$; $2,\overline{11}$; $\sqrt{0{,}081}$; $\frac{\sqrt{20}}{\sqrt{5}}$.
3. Prove that the number $\sqrt{2}$ is irrational.
4. Which of the numbers $\sqrt[5]{5}$, $\sqrt{2}$ is greater?

5. Remove the square root of the denominator of the fraction $\frac{1}{\sqrt{3}-\sqrt{2}}$.

   The result is a) integer

   b) rational number

   c) an irrational number

6. Edit the expression $\sqrt{a^6b^4 - a^4b^6}$, at which $a, b \geq 0, a > b$.

Results and analysis of test

The test results are processed from the responses of students.

Test

1. List the prime numbers from 1 to 30. Make of them irrational numbers.

$K -$ the set of prime numbers

Answer	Number of responses
1.1 : $K = \{1,2,3,5,7,11,13,17,19,23,29\}$ Irrational numbers: $\sqrt{2}, \sqrt{3}, \ldots \sqrt{29}$	7
1.2 : $K = \{1,3,5,7,11,13,17,19,23,29\}$ Irrational numbers: $\sqrt{1}, \sqrt{3}, \ldots \sqrt{29}$	6
1.3 : $K = \{2,3,5,7,11,13,17,19,23,29\}$ Irrational numbers: $\sqrt{2}, \sqrt{3}, \ldots \sqrt{29}$	17
1.4 : $K = \{1,2,3,5,7,11,13,17,19,23,29\}$ Irrational numbers: $\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots \sqrt{29}$	25
1.5 : $K = \{1,2,3,5,7,11,13,17,19,23,29\}$ Irrational numbers: can not be made	4

2. Decide which of the following numbers are rational and the irrational: $\sqrt{0,25}$; $1,213\overline{213}$; π; 3,21875143 ...; $\sqrt{150}$; $\sqrt{900}$; $\sqrt{0.28}$; $\sqrt{25}$; 7,3284317 ; $2,\overline{11}$; $\sqrt{0,081}$; $\frac{\sqrt{20}}{\sqrt{5}}$.

Answer	Number of responses
2.1 : correctly, reasoned	19
e.g. π - has not completed decimal developing – the irrational number $\sqrt{900} = 30$ it is the rational number $2,\overline{11}$ - it is periodic - the rational number	
2.2 : they did not know edit the number under the square root	13
2.3 : estimated, without justification	17

2.4 : they did not know to divide the numbers 10

3. Prove that the number $\sqrt{2}$ is irrational.

Answer Number of responses

3.1 : by using the Pythagorean Theorem 11

$$\sqrt{2} \in I \quad c^2 = a^2 + b^2$$
$$c^2 = 1^2 + 1^2$$
$$c^2 = 2$$
$$c = \sqrt{2}$$

3.2 : $\sqrt{2} = \frac{p}{q} \qquad /()^2$

$$2 = \frac{p^2}{q^2} \qquad p, q - \text{ relatively primes}$$
$$2q^2 = p^2 \qquad 2/p \quad \wedge \quad p = 2a$$
$$2q^2 = 4a^2q^2$$
$$q^2 = 2a^2 \qquad 2/q \ .$$

15

3.3 : by using calculator 8

$\sqrt{2} = 1{,}41 \ldots$

3.4 : they did not know 25

4. Which of the numbers $\sqrt[5]{5}$, $\sqrt{2}$ is greater?

Answer Number of responses

4.1 : $\sqrt[5]{5} = 5^{\frac{1}{5}} = 5^{\frac{2}{10}} \qquad \sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{5}{10}}$

$$\sqrt[10]{5^2} \quad ? \quad \sqrt[10]{2^5}$$
$$25 < 32 \quad \text{then} \quad \sqrt[5]{5} < \sqrt{2}$$

5

4.2 : estimated, without a solution 24

4.3 : by using calculator 9

$$\sqrt[5]{5} \quad ? \quad \sqrt{2}$$
$$1{,}379 < 1{,}4142$$
$$\sqrt[5]{5} < \sqrt{2}$$

4.4 : they did not know 11

5. Remove the square root of the denominator of the fraction $\frac{1}{\sqrt{3}-\sqrt{2}}$.

The result is a) integer

b) rational number

c) an irrational number

Answer Number of responses

5.1 : $\frac{1}{(\sqrt{3}-\sqrt{2})} \cdot \frac{(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})} = \frac{\sqrt{3}+\sqrt{2}}{3-2} = \sqrt{3} + \sqrt{2} \in I$ 23

5.2 : $\frac{1}{(\sqrt{3}-\sqrt{2})}\cdot\frac{(\sqrt{3}-\sqrt{2})}{(\sqrt{3}-\sqrt{2})}=\frac{\sqrt{3}-\sqrt{2}}{3-2}=\sqrt{3}-\sqrt{2}\in I$ 13

5.3 : $\frac{1}{(\sqrt{3}-\sqrt{2})}\cdot\frac{(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})}=\frac{\sqrt{3}+\sqrt{2}}{9-4}=\frac{\sqrt{3}+\sqrt{2}}{5}\in I$ 4

5.4 : $\frac{1}{(\sqrt{3}-\sqrt{2})}\cdot\frac{(\sqrt{3}+\sqrt{2})}{(\sqrt{3}+\sqrt{2})}=\frac{\sqrt{3}+\sqrt{2}}{3+2}=\frac{\sqrt{3}+\sqrt{2}}{5}\in I$ 10

5.5 they did not know 9

6. Edit the expression $\sqrt{a^6b^4-a^4b^6}$, at which $a,b\geq 0, a>b$.

Answer Number of responses

6.1 : $\sqrt{a^6b^4-a^4b^6}=\sqrt{a^4b^4(a^2-b^2)}=\ a^2b^2(a^2-b^2)^{\frac{1}{2}}$ 21

6.2 : $\sqrt{a^6b^4-a^4b^6}=\sqrt{a^4b^4(a^2-b^2)}=\ a^2b^2\sqrt{(a-b)(a+b)}$ 14

6.3 : $\sqrt{a^6b^4-a^4b^6}=a^3b^2-a^2b^3=\ a^2b^2(a-b)$ 13

6.4 : $\sqrt{a^6b^4-a^4b^6}=a^{\frac{6}{2}}b^{\frac{4}{2}}-a^{\frac{4}{2}}b^{\frac{6}{2}}=a^3b^2-a^2b^3$ 4

6.4 : they did not know 7

On the chart (Figure 1) we can see the number of correct answers to each question.

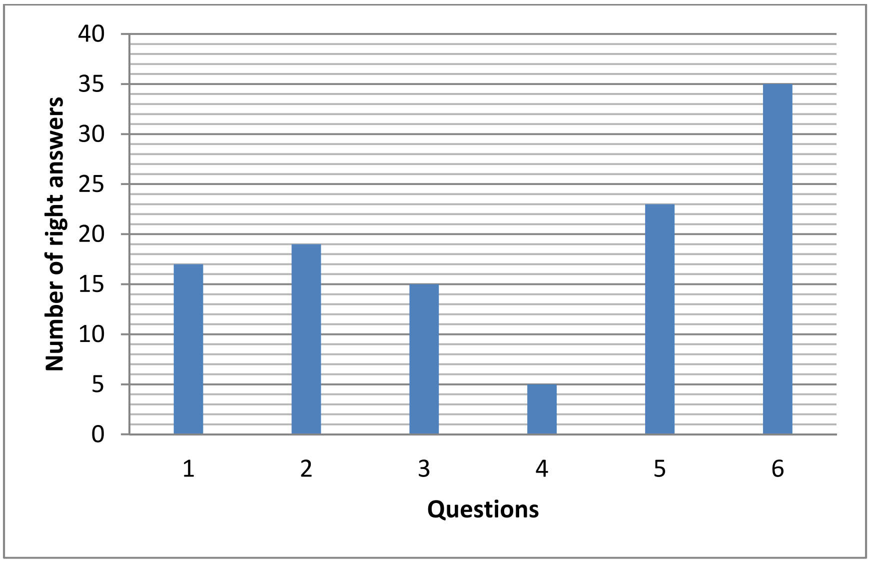

Figure 1: Quantification of right results

Discussion of results

In these responses we identify several overlapping themes: applications of an incorrect or incomplete definition, distraction by a calculator display and lack of awareness of the relationship between fractions and repeating decimals. By testing on understanding irrational numbers we concluded that students relied on decimal representations rather than common fraction representations. We further observed that the connection between two ways to describe irrational numbers - as lacking representation as a ratio and having infinite non – repeating representation – were not well understood. Examples play an important role in learning mathematics. It is hard to imagine learning math without the consideration of specific examples. Very good way is a “Problem solving”. What is mathematical problem-solving? Mathematical problem - solving is the heart of any mathematicians work and to become a mathematician, one who discovers, conjectures, tests and proves one must become a problem solver. For this to happen, students must therefore engage in solving real problems. Problem solving has a special importance in the study of mathematics. A primary goal of mathematics teaching and learning is to develop the ability to solve a wide variety of complex mathematics problems.

Conclusion

We investigated the understanding of irrational numbers of the group of students at the high school. In this report we focused on the role that representations play in concluding rationality or irrationality of a number. We discovered, that

- the definitions of irrational, as well as rational, numbers were not “active” repertoire of their knowledge;
- there was a tendency to rely on a calculator
- there was a confusion between irrationality and infinite decimal representation, regardless of the structure of this representation

A general suggestion for the teaching practice calls for a tighter emphasis on representations and conclusions that can be derived from considering them. In particular, attending to the connections between decimal and other representations (geometric, symbolic, common fraction, even continued fractions) of a number can be an asset.

Mathematics is a specific subject. To have a number of relevant information, to know the facts or isolated terms, to recall mathematical sentences, definitions is only the first step. The cardinal necessity is to understand their mutual context, which is the only source of true knowledge also applicable in reality.

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PRE-SERVICE TEACHERS' PROBLEM POSING IN COMBINATORICS

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ABSTRACT. *Combinatorics is seen as one of the more difficult areas of mathematics to teach and to learn. Mathematical knowledge for teaching combinatorics of 14 pre-service teachers for primary school developed during session integrating mathematical and pedagogical activities was assessed through the problem posing. Knowledge of combinatorics of some was enhanced but not in satisfactory extend. Lack of subject matter knowledge influenced students' ability to pose and subsequently solve combinatorial problems.*

KEY WORDS: *teacher education, combinatorial thinking, mathematical knowledge for teaching*

CLASSIFICATION: *K209*

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Introduction

Discrete mathematics (including combinatorics and graph theory) became part of NTCM standards in United States in 1989 [1]. Since then, the combinatorics is moving to lower grades also in Europe, e.g. in 2004 in Germany (Bayern) [2], in 2007 in Portugal [3]; and since 2008 in Slovakia [4]. Future teachers of mathematics did not experience approach suitable for younger pupils as learners, therefore special focus should be put on this area of mathematics in education of future teachers.

It was shown [5] that when given enough time and hands-on problems, even usually low achieving students can do well in solving combinatorial problems, and vice-versa usually high-achieving students can lose track when dealing with novel problems in combinatorics. Lockwood [6] stressed the role of set of outcomes also in solving of advanced combinatorial problems by tertiary students. She found that students deriving the expression/formula directly from the wording of the task are more likely to overestimate the number of configurations required in task comparing to students first listing a few elements of the set. Hejný [7] claims that development of combinatorial thinking should be based on appropriate amount of combinatorial situations which pupil should deal. By combinatorial situation he understands the triplet: base set (elements inputting to configurations), set of outcomes (set of configurations satisfying the conditions of the task) and organizational principle (structure of set of outcomes in sense of Lockwood). In study [8] it was found that the most efficient verification strategy of students for combinatorial problem is to solve it in other way.

Thus the teacher has to have appropriate knowledge to organize the content and the lesson. Furthermore, he/she should be able to track the process of development for his/her pupils. Jones et al. [9] identified stages in development of combinatorial thinking of children based on SOLO model [10]: Level 1 (Subjective): listing elements in random order, without looking for systematic strategy; Level 2 (Transitional): use of trial-error strategy, discovery of some generative strategies for small sets of outcomes; Level 3 (Informal quantitative): adopting generative strategies for bigger sets or three- and more-

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dimensional situation; Level 4 (Numerical): applying generative strategies and use of formulas; Level 5 (Extended abstract): generalization of relations.

Mathematical Knowledge for Teaching

Issue of teacher knowledge was first arisen in work of Shulman [11]. It was further developed [12, 13] and related to teachers' practice [14] by research team about Deborah L. Ball. Mathematical knowledge for teaching (MKT) is understood as knowledge going behind standard use of mathematics methods, it includes how to represent mathematical concepts and procedures to students, explain mathematical concepts to students and analyze students' solutions and explanations [12].

MKT consists of two domains: subject matter knowledge (SMK) and pedagogical content knowledge (PCK). Both of these are further divided into three subdomains. SMK consists of: (1) Common content knowledge (CCK) refers to general knowledge of mathematics; (2) Specialized content knowledge (SCK) is specific to mathematics teaching. It is used when students' solutions, explanations and reasoning are assessed; (3) Knowledge at the mathematical horizon (KMH) which means relations between concepts and topics included in the mathematics curriculum.

PCK is divided into: (1) Knowledge of content and students (KCS) means understanding students' mathematical thinking; (2) Knowledge of content and teaching (KCT) deals with the ability of teacher to choose and arrange suitable problems for the classroom; and (3) Knowledge of curriculum (KC).

Domains of MKT are related one to each other. Furthermore, "mathematical experiences and pedagogical experiences cannot be two distinct forms of knowledge in teacher education" [15, p. 1964]. Integrated approaches help students to "get a broader view of mathematics, to see its relevance to teaching and to recognize the need for their mathematical and pedagogical development." [15, p. 1964]

Research questions

Can be knowledge in combinatorics developed simultaneously with pedagogical content knowledge?

What kind of knowledge influences primary pre-service teachers' ability of posing combinatorial problems?

The study

Participants of the study were members of international group of 14 pre-service teachers for primary schools taking part in EPTE program attended the session about problem-solving in combinatorics within the mathematic module. The session was led by one of the authors of the paper. We will consider her as participant-observer. The session was audiotaped to enable further analysis. According to [16] "problem posing provides an opportunity to get an insight into natural differentiation of students' understanding of mathematical concepts and processes and to find obstacles in understanding and misunderstandings that already exist", so the data analysis is focused mainly on problem-posing part of the lesson.

Description of the session

The session started with group-work, 14 students formed 5 groups. In the analysis we use notation S*x*G*y* for student *x* from group *y* and T for teacher. Beside problem-solving activities for developing CCK and SCK, students assessed pupils' solutions and pose problem suitable for primary school to connect their subject matter knowledge with pedagogical content knowledge.

The first problem aimed to estimate the level of combinatorial thinking of participating students. It was only two-dimensional: 'Four friends met and shook their hands. How would you describe all handshakes?' The wording of the problem was intentionally formulated to describe the handshakes, to make students to choose the representation of the set of outcomes and not to solve the problem by formula/expression. Only after solving this part, the second question 'How many handshakes there were?' appeared.

Three of five groups were able to solve the problem by employing generative strategy (level 2). Two of successful groups used diagrammatic representation, one group chose a table. Only one student was able to use the formula/expression without hint of the teacher (level 4).

Overall low level of combinatorial thinking of participating students could be seen. Even the presenters from the groups which solved the problem encountered difficulties during the presentation of their own solution. They were not able to explain in detail their employed strategies, particularly why they chose them.

After finishing the frontal discussion about the problem, levels of combinatorial thinking according [9] were introduced. Different students' solutions served as examples for the levels. Second problem [17] was again solved within the group-work. 'How many text messages are sent if four people all send messages to each other? How many text messages are sent with different numbers of people? Approximately how many text messages would travel in cyberspace if everyone in your school took part? Can you think of other situations that would give rise to the same mathematical relationship?' Students were asked to solve this problem by listing elements of set of outcomes, giving it appropriate structure and by formula. All groups were able to solve this problem, mostly by realizing that the number of text messages was twice as much as handshakes in previous problem. After the frontal discussion excerpts of pupils' work also included in [17] was given to students to assess it and possibly formulate an advice which they will give to pupil in their future class if he or she will come with solution like those in the set. They were also asked to come up with good questions which can lead pupils to higher level of solution.

Insufficient level of PCK can be seen in the reflection of one participating students on the session: *Some of them [strategies] were very easy to reproduce, some of them not. Especially when they had a mistake in their strategy and thinking it was not so easy to follow their ideas.*

The third problem was three-dimensional and table or graph was not suitable structure for set of outcomes. 'Peter spends too much time with the computer, so his parents decided to use the password. Peter heard that the first password consists of 4 characters, digits 0 to 3, each only once.' [18] Four groups succeeded in solving this problem, mostly by ordering the strings by their value as numbers. One group was already familiar with tree diagram, which was also presented as possible generative strategy suitable for more-dimensional combinatorial problems. Just after they solved the problem by expression/formula $4 \times 3 \times 2 \times 1$ students realized that this was factorial of number 4. So they activated their CCK, although only on the level of visual recognition, without deeper understanding of the matter. Students' understanding of the importance of generative strategies can be seen: *Of course my chaos system failed and the "tree-system" was quite better to understand.*

The last activity within described session was to design a combinatorial problem suitable for primary school. Students were asked to find the solution by listing elements of set of outcomes, suggest appropriate structure for set of outcomes and, finally, solve the task using the formula/expression. Outcomes of this group-work and following discussion are described and analyzed in following section.

Data analysis

Group 1

Posed problem: Silvia has 5 skirts (yellow, red, green, pink, blue), three pairs of shoes (sneakers, ballerina shoes, slippers) and 2 tights (orange and brown). How many outfits can she create?

Group was able to use the multiplication rule in this problem and estimate the number of outfits as $5 \times 3 \times 2 = 30$. They have chosen tree diagram as the structure for set of outcomes. They posed and solved the problem very quickly and then they discussed possible attitudes how to present the solution in the classroom.

Group 2

Posed problem: The cipher bicycle lock has four digits 0-9. You forgot the password. How many numbers (in worst case) you have to try to unlock it?

The group struggled with the solution. When the teacher came and asked how their work was going, students replied that they got lost. Teacher solved the task in her mind and found out that 104 are too many possibilities for primary level. She thought that students are able to solve the designed task by expression/formula and she wanted them to come up with the task which has set of outcomes of reasonable size.

Teacher (T): *How many possibilities there are? Can you calculate it without writing them all down?*

S1G2: *OK, let's have only numbers 0-4*

There is not clear, whether the student was aware of the solution of the problem or she lowered the number of possible digits based on teacher's question. Then she started to write down (see Table 1):

0,1,2,3	24 possibilities
1,2,3,4	24 possibilities
0,2,3,4	24 possibilities
0,1,3,4	24 possibilities
Total	96 possibilities

Table 1 Excerpt of solution of group 2

S1G2: *I miss some, but I do not know how they should look like.*

T: *Why do you think the numbers cannot repeat?*

Student wrote down the group of four digits

0,0,1,3

S1G2: *How can I find out how many of this kind there are?*

Another student from this group did not participate in discussion with the teacher, but after the first student lowered the size of base set, she tried to solve the problem on her own.

S2G2: *Let it be, we will use the tree.*

Then the group drew one "strand" of the tree diagram, the codes starting by 11. During the presentation they commented on it:

S2G1: *it is obvious how the tree will look like*

S1G1: *and the expression will be 5 times 5 and so on.*

After the two groups presented their problems, teacher wanted the students to compare the two problems, both solved by tree diagram and multiplication rule.

T: *How do the two presented problems differ?*

S1G4: *There are so much other possibilities in the second case.*

S1G1: *I have chosen quite small numbers to have it feasible to draw whole the tree on the blackboard.*

S2G3: *In the first case, you know how many pieces of each kind you have, but in the second task you have to think after each level of the tree.*

S2G1: *Yes, in our case you just multiply the number of things. But in the second task you multiply 5×5×5×5, you have to come up with the next number to multiply.*

Group 3

Posed problem: You have 5 children (Michal, Fero, Juro, Katka and Zuzka) and five pieces of fruit (banana, apple, orange, pear and a kiwi). How many possibilities you have to give children the fruits?

The group struggled with the solution; they tried to find all the permutations of the set of children and all permutations of the set of fruits. One member of group calculated that number of possibilities how to arrange children is 5! = 120, as well as the number of possibilities how to arrange fruits, but she was not able to put the two obtained numbers together. So, they decided to solve the task on lower level.

They started by drawing the diagrams mapping children and fruits together. After 3 diagrams they refused from drawing and started to write down some configurations of ordered pairs. There were app. 20 listed on their paper when teacher came to check the progress in the group. She scanned solution of the group but neither saw any structure of the set, nor knew the wording of the problem to analyze where the trouble came from.

T: *What problem have you posed?*

S1G3: *We have children and we have fruits: banana, orange, apple, pear and kiwi, so we are going to give out the fruits to children.*

Teacher got a bit confused and wanted students to elaborate more the wording of designed problem.

T: *So, you can give bananas to all the children.*

S2G3: *No, you have only one banana.*

T: *Does it mean that each child will get different kind of fruit?*

S1G3: *Yes, exactly.*

T: *And what about the number of children and fruits, are they the same?*

S2G3: *Yes, we have 5 children and 5 pieces of fruits.*

Teacher checked down listed configurations and chosen two that differed only on order, but she was still not sure about the wording of the problem.

Mb, Fa, Jo, Kp, Zk   Fa, Kp, Mb, Zk, Jo

T: *How are these two groups different?*

S1G3: *In the left case, Michal is the first to get fruit, in the right one Fero is the first.*

T: *So, did you include this in your formulation of the problem?*

S1G3: *No, I just want to give children the fruits.*

T: *Does it matter, in your combinatorial situation, who get the fruit first?*

S1G3: *No… so, I can put the fruits in one front and mix the children. Who comes first, will get a banana, the second will get apple.*

T: *Can you continue with your task now?*

S1G3: *There are too many possibilities…* [She looked at their expression to check how many items she will need to avoid too many possibilities]. *I will have only three of the children and the fruits.* [Then she wrote table with all the possibilities]

Students independently realized that listing of 120 possibilities was not good idea for the classroom. Using the mathematical knowledge she chose the appropriate base set
T: *How did you obtain the columns of the table?* (see Table 2)

Banana	Fero	Fero	Michal	Michal	Juro	Juro
Apple	Michal	Juro	Juro	Fero	Michal	Fero
Orange	Juro	Michal	Fero	Juro	Fero	Michal

Table 2: Excerpt of solution of group 3

S1G3: *I chose the first, and then there are only two possibilities for other two. So I write the two remaining children and then just switch their order.*

The system that student chose is not a generative strategy. That wouldn't be usable with many items. So she did not think about it, just adjusted it to get the outcome. On the other side, the student was already aware of the number of possibilities and she could consider that the sophisticated structure was not necessary for such a small set. After being satisfied with the mathematical success, she might not put the priority to solution on lower level, suitable for primary pupils.

Conclusions and discussion

In developing PCK, also CCK has to be taken into account. Students in groups 2 and 3 intuitively underestimated the number of possibilities in cases of variations and permutations what is in accordance with [19]. During process of solving the posed problem both groups lowered the size of base set, but in case of group 2 not enough. Furthermore, group 2 did it only after prompting by the teacher. The difference can be in the level of combinatorial thinking of students. Although group 3 struggled with solution in solving their problem, they got partial result in form of expression/formula. After teacher's intervention in set of outcomes, they were able to solve the posed problem. Group 3 decided about the number of children and fruit by expression/formula, group 2 did not calculate the number of possibilities, even during the presentation they said *you have to just multiply 5 times 5 etc*. Student from another group (group 1) had to remind them with the number. High level of CCK of students in group 1 was also observable in the comment about informed decision when posing their problem. We can assume that without satisfying level of subject matter knowledge students can experience difficulties in activities aiming to pedagogical content knowledge.

The role of CCK, especially in pre-service teacher training, is often underestimated, particularly by students. During problem-posing and subsequent problem-solving of posed problem the importance of mathematical knowledge may emerge also for student-teachers. According to [16] and [20] problem-posing activities are very suitable in training of mathematics teachers. They can uncover possible misconceptions or insufficient knowledge of student-teachers. Combinatorics was not new knowledge for participating students; all the students came up with factorial as a formula for number of permutations after solving the Internet problem by tree diagram. Even after passing courses in mathematics it seem advantageous to include activities developing the mathematical knowledge of student-teachers in courses focused on pedagogies. We can see the shift in the level of combinatorial thinking in the comment of student from group 2 [18] *Of course my chaos system failed and the "tree-system" was quite better to understand*, although even after the shift it was not adequate for future teacher. On the other side, to experience own development as a learner can enhance both, subject matter knowledge and pedagogical content knowledge. This is in accordance with [16], [15] or [21] claiming that mathematical and pedagogical knowledge should be blended to develop mathematical knowledge for teaching.

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EVALUATION OF A QUESTIONNAIRE CONCERNING WORD PROBLEMS CREATING

EVA MOKRÁŇOVÁ

ABSTRACT. *In Mathematics Education seminar we conducted a research probe mapping the ability of future mathematics teachers to create their own word problems in the relevant context. The article offers the evaluation of the questionnaire, which was consequently given to the students - future teachers of mathematics.*

KEY WORDS: *work problems, mixture, research probe, questionnaire evaluation*

CLASSIFICATION: *F90*

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Introduction

We often hear the opinion that the connection of the real world and the mathematics teaching is important and necessary. This connection is often superficial in the classrooms. From our point of view it is caused by the following reasons:

- Math teachers don't have enough materials, textbooks and collections of mathematical tasks (mathematical tasks may be simple equations, word problems, or measurement activities) that support the inclusion of situations from everyday life into mathematics teaching.
- Teachers are not able to integrate the problems of everyday life to mathematics teaching.
- Students' knowledge is not sufficient for everyday life complex problem solving.
- Many tasks are referred to as "pseudo-realistic". Often neither context nor the data used are real [3].
- The task context is not interesting for students.
- Real-world mathematical tasks usually cannot be directly applied into teaching; they have to be modified due to the pedagogical and practical reasons.

We realize that there might be more reasons than we have mentioned. For this reason, we believe it is important that future mathematics teachers are not only able to solve problems with real context, but also to create them. They should be prepared for this task during their university studies.

The research problem

In the Didactics of Mathematics seminar, we mapped the ability of the second year Master's students of Mathematics Education to individually create word problems with real context.

In 2010, we carried out the first research probe concerning the creation of word problems on the uniform linear movement with a realistic context. The results were presented on Zilina Didactic Conference with international participation didZA [2].

The second research probe was carried out in 2012 on eight students. The aim of the seminar was to summarize the methodology of word problems, to show to students the aspects of the formation of word problems with real context and subsequent independent work of students. The students' task was to create four work word problems and four word problems about mixtures from everyday life.

From created tasks we composed two databases. The first database contains 32 work word problems and the second database 32 word problems on mixtures. At the next seminar, each student addressed four work word problems and four problems on mixtures selected from these databases so that no student got his own task. For each task students received a questionnaire for the task assessment. The seminar was attended only by seven students so there were only 28 questionnaires returned.

The questionnaire includes questions from three different areas:

A. The task´s context

B. Intelligibility

C. Numerical data

Evaluation of the questionnaires:

A. The task´s context

1. What is the context of word problems?

As we have already mentioned, many word problems in textbooks and in task collections are "pseudo - real". For this reason, in the first question we examined what context were the tasks created by students associated with. In the second question we wanted to find out if the created tasks come from everyday life.

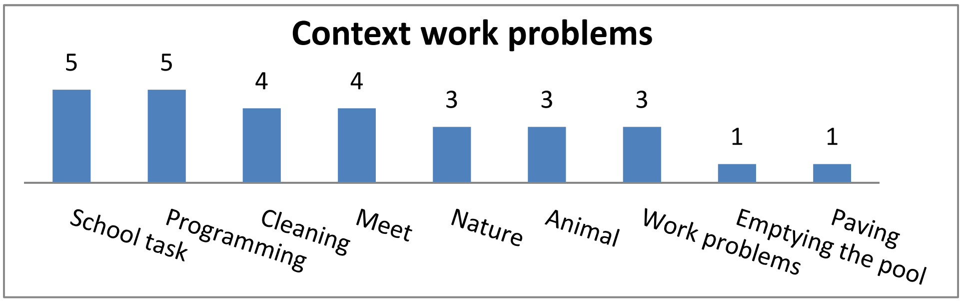

Figure 1

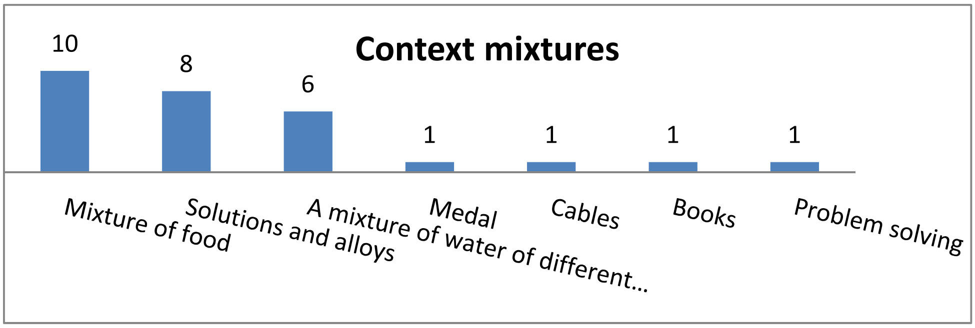

Figure 2

2. The word task has:
a) Context from real life
b) Abstract nature

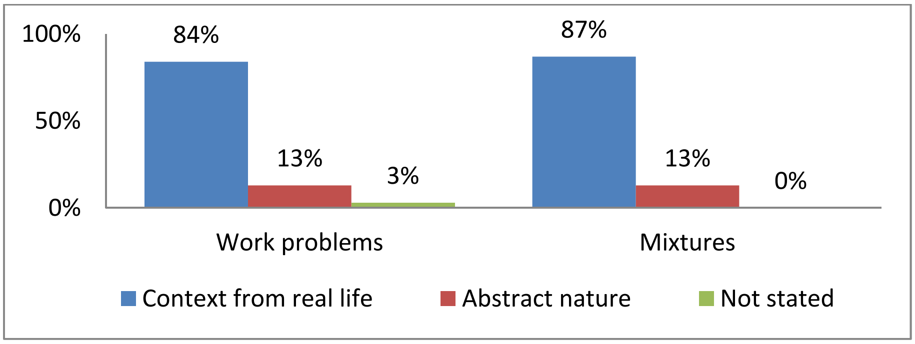

Figure 3

In spite of the fact that the students' task was to create word problems with real context, 13 % of word problems in both databases has abstract character.

B. Intelligibility

3. Assignment of the word task is:
a) Understandable
b) Partially understandable
c) Rather unclear
d) Partially unintelligible
e) Unintelligible

We often see that tasks are unintelligible and unclearly formulated for students. Therefore, we think that creators of word problems should have the created word problems solved by at least one student in order to eliminate possible errors in the formulation.

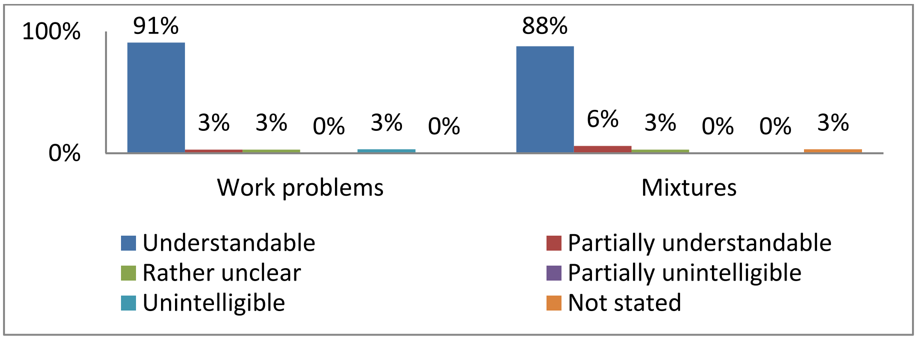

Figure 4

Students who considered the tasks to be partially unintelligible or unclear could reformulate them to be understandable. Despite this, students reported that 3 % of the tasks seemed unintelligible; none of them had reformulated them.

4. The number of sentences in word problems.

Students generally prefer to solve tasks with briefly and clearly specified task´s conditions and questions. In lengthy texts students lose the meaning and do not know what to solve.

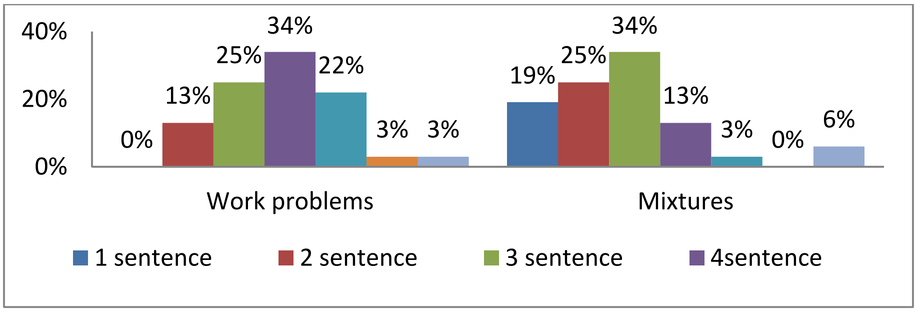

Figure 5

5. The number of questions in word problems.

The research that we have carried out on a sample of 9th grade students shows that if the task has more questions, students often respond only to the first one and forget to solve the second one.

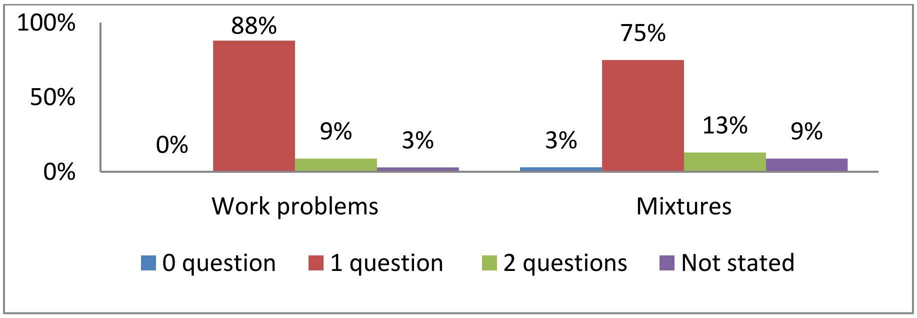

Figure 6

6. Task is given:

a) By imperative

b) By question

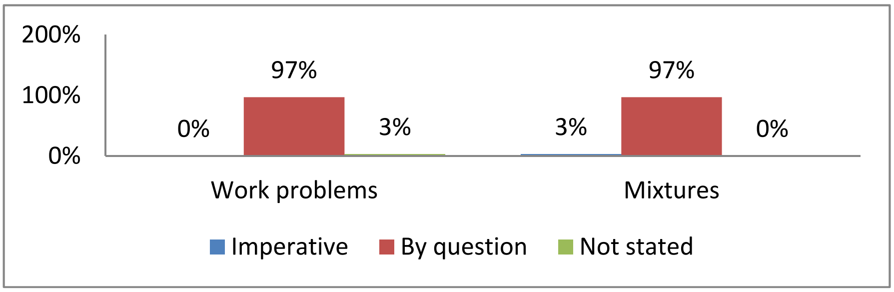

Figure 7

C. Numerical data

7. Numerical data in the assignment of word problem are real:
a) Yes
b) No
c) I do not know

In this question we examined whether the data used in the assignment are real, because the created context is often real, but the data are not.

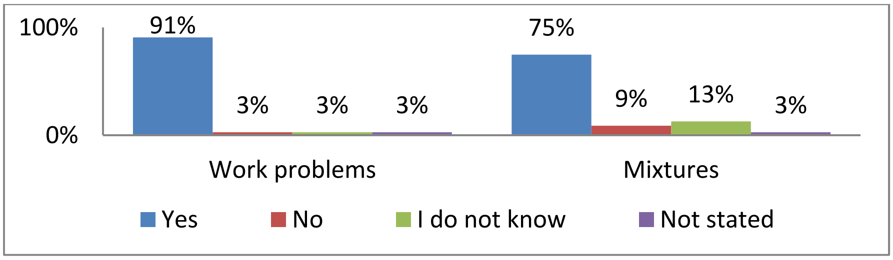

Figure 8

8. The number of numerical data in word problem leading to numerical result is:
a) Sufficient
b) Insufficient
c) Word problem includes extra data.

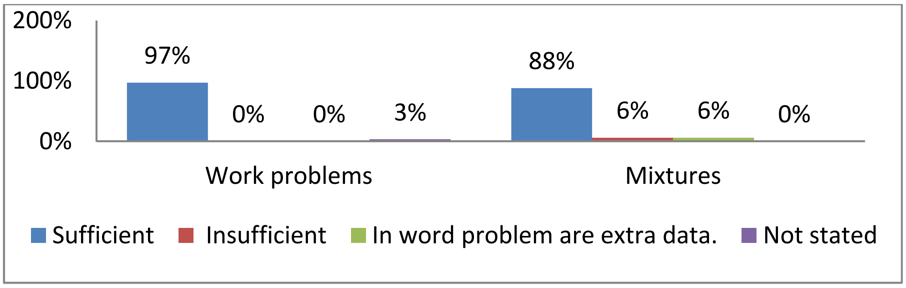

Figure 9

9. Numerical result of a word problem is:
a) Integer
b) Fraction
c) Decimal number

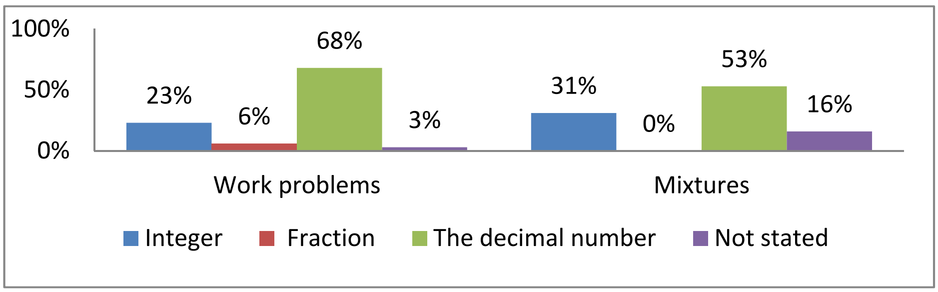

Figure 10

68% of students who have solved the work word problems stated the result in decimal form, in case of mixture tasks it was 53 %. Such result is often rounded, although students interpret results of tasks more often as a decimal number then as a fraction, as it is more precise and concise.

10. At this point we left to students the space for their comments on word problems.
"It is not a work word problem"
"I can't get the result."
"Difficult."
"The task has no logical solution; prices should be given in reverse."
"The result contradicts the conditions of the assignment."
"The result does not make sense; we do not buy" negative "quantity of goods."
"The word problem is more likely for the better students."

Conclusion

Currently, math teachers are forced to create their own materials and teaching tasks. Our effort at the seminar was to present the issue of creating work word problems and word problems about mixtures, as well as to offer a methodology of word problems creation to mathematics education students.
Despite the fact that the students identified the created problems as having a real basis in the questionnaire, many of them were "pseudo-real". The tenth section of the questionnaire shows that students often did not approach the tasks creation responsibly and were not solving the word problems they did not create.
Nevertheless, we think that within the Didactics of Mathematics it is necessary that students create their own databases with word problems that might be useful for them in their future role as teachers in schools.

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ACTA MATHEMATICA 17

THE DEVELOPMENT OF THE CONCEPTS ABOUT SIMILARITY AND RATIO IN MATHEMATICS EDUCATION

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ABSTRACT. *In our article we focus on the connections between some topics of mathematical curriculum at all level of primary and secondary school. These topics are similarity, homothety, ration and fractions. We mention about the motivation and the creativity in problem posing process as an important aspect of education. The creation of the tasks which are connected with one picture on different level of education is presented as an example of mentioned processes. We used a photo of geometrical garden as a motivation tool. We wanted to show one of the possible approach, how to use real life object for the creation of the geometrical problems in various levels of difficulty.*

KEY WORDS: *ratio, similarity, geometrical task, education*

CLASSIFICATION: *G10, D50*

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Introduction

A subject matter like similarity is one part of the mathematical education since the primary school level to secondary one. The process of enlargement and reduction of geometrical shapes and figures begins as a square grid tasks and continues using the homothety. There is a strong connection among the similarity as the geometric topic; and ratio and fractions as the arithmetic topic in the school mathematical education.

As is mentioned in [1] the interdependence of these topics and the relationship among different parts of the mathematics (the arithmetic, geometry and algebra). The author pays an attention to the fact, that the geometrical and the numerical context are interlinked. Also notes that many topics follow each other without the reminder and this fact if often omitted in school praxis. The author compared mathematical topics starting the number comparisons through the fractions and the ratios to the proportions. The author defines the *k* factor to define the Identity, the Similarity and the Homothety.

In next sections we focus on three areas:

- the similarity, homothety, ratio and fractions in the mathematical education at primary and secondary school,
- the motivation and the creativity in the problem posing process,
- creation of the tasks connected with a picture on different level of education.

The similarity and the ratio in the mathematical school education

According to The National Educational Program in Slovakia pupils meet with propaedeutic of fractions and similarities already at the primary level of education. As is stated in [2], pupils should acquire concepts as a whole, part of a whole, the number of equal parts (the division), the group size (after division) and have the ability to divide a whole into equal parts, to divide according to content. The curriculum of geometry of the

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primary and lower secondary education included topics such are enlarging and reduction of geometrical shapes in a square grid which is connected with the ratio. The curriculum according to [3] includes the topics: ratio; to divide unit in a given ratio; plan and map scale; continual and reciprocal proportion; simple rule of tree (plicate too); utilization of continual proportion in practice (contextual and stimulative tasks). Pupils are working with similarity in the topics: similarity; similarity of triangles; similarity of geometric figures; ratio of similarity; division of the segment in a given ratio; solution of a mathematical (numerical) and constructions tasks; utilization of similarity by measuring of heights and distances; topographic works in the real situation. This part of geometry is developed at the higher level of secondary education according to [4] in the topics: practice mathematics – map and plan scales; basic geometric figures; geometric points of places (constructions); measuring; estimating; geometry of acute angle; identity; similarity. According to the performance standards in this domain, the student should know: use the rule of three; use the continual and reciprocal proportion to solve simple practical problems; construct in simple cases the basic planar figures; use right triangle geometry to calculate the size of angles and lengths of sides; to solve applications tasks by using trigonometry; determine similarity of the triangles; apply the relationship between similar triangles to solve geometric problems; deduce Pythagorean Theorem and Euclid sentences; count the length and distance by using these sentences; determine the approximate dimensions of unavailable services by using similarity.

The motivation and the creativity in the problem posing process

Mathematics is an integral part of the real-life not only for many daily activities but also for a wide variety of work situations. It is necessary to transfer the math knowledge and skills gained in schools to the real-life that require the individual to reason, calculate, estimate or apply math knowledge to solve real-life problems and also to communicate mathematically [5].

There are a lot of objects in our ordinary life which could be used for creation challenging learning environment and develop pupils and also teacher´s creativity in mathematics. As is in [6] described, mathematical creativity as an ability to analyze a given problem from a different perspective, see patterns, differences and similarities, generate multiple ideas and choose a proper method to deal with unfamiliar mathematical situations.

According to survey [7], creativity of children in great deal depends on the teacher's approach. When pupils solve only standard tasks by using always the same methods, they have problems to change their learning way or create a task independently. Children can develop certain commodity in thinking, little initiative or even unwillingness to work.

Creativity in the mathematics classroom is not just about what pupils do but also about what we do as teachers. If we think creatively about mathematical experiences that we offer our pupils, we can open up opportunities for them to be creative [8]. Creative teaching requires from teacher to create and exercise such tasks, which would enable pupils to use their acquired knowledge more freely, in new contexts and when solving new and unknown problems [9].

Motivation and creativity in the geometrical problem posing process is processed in the work of several authors, for instance [10], [11], [12], [13].

The Similarity and the Homothety in the tasks

We will show the way how to use one picture for the creation of geometrical problems at various levels of education. We apply graded approach and present the tasks based on the school curriculum and with real live context. The motivation was found in the nice geometrical gardens (Fig.1) and we used the photo for drawing elementary pattern in the square grid (Fig. 2).

Fig. 1: The geometric patterns in Anguri Bagh garden, Agra Fort. Agra in India.

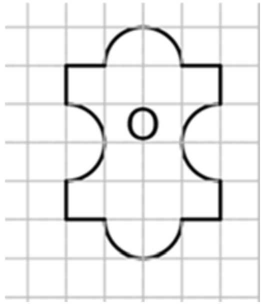

Fig. 2: The geometric patterns O in the square grid

Task at the primary school level

Primary school pupils should solve the task given in the square grid, to draw the pattern which will be two times bigger. We will use twice the size unit of a square grid and pupils can only retrace individual lines in one square. Result will be two times increased pattern O_1. Pupils can draw or use the drawing tools: the ruler and the compasses.

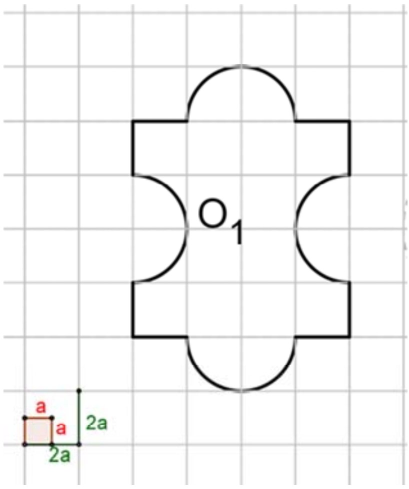

Figure 3: Solution in the twice the size square grid

Task at the lower secondary school level

At the lower secondary school level we can solve the previous task, but in the same square grid (the same unit square). Pupils have to pay attention to the individual elements in the pattern and each of them draw twice as large. It is important to determine the center and the radius of a circle (Fig. 4).

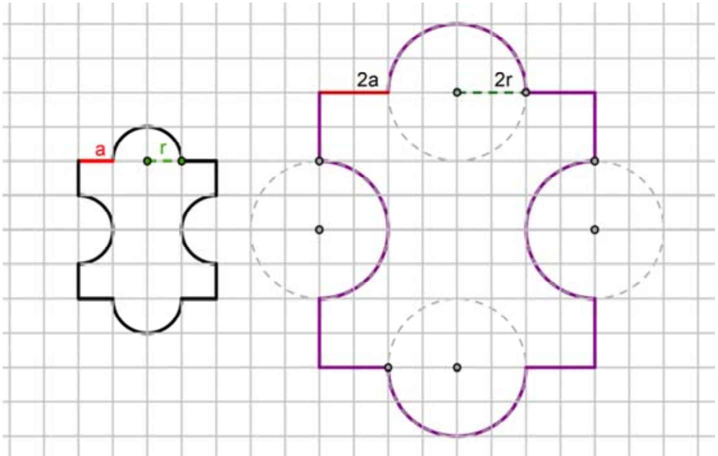

Figure 4: The solution in the same square grid

Task at the higher secondary school level

At the higher secondary school level we can use the homothety with scale factor *k* equals 2. We can also create opposite task, to find a center and a scale factor of the enlargement that will transform a pattern O onto a pattern O_2 (Fig. 5)

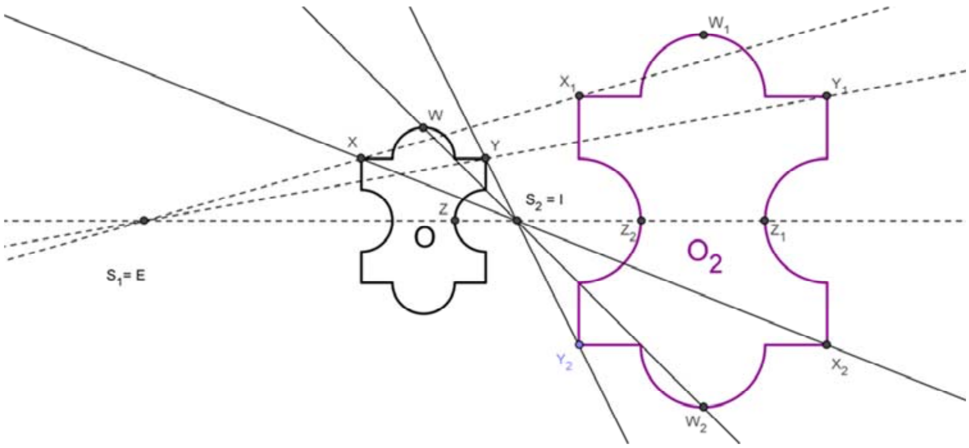

Figure 5: Application of homothety

Every mentioned geometrical problem is connected with using the ratio. We can construct, for example, tree times bigger figure and then two times smaller figure and search a similarity ratio between first and third figure. We can create a word geometrical tasks with various context to all of previous examples. The pupils can work in a role as a garden architects or a puzzle creator, can construct the same pattern in different sizes, create wall or floor tiles, do a patchwork and so on. Work with non-typical geometric shapes could be more demanding for analyzing of the individual shape elements, precision, patience, construction, but good motivation and connection with real life objects can diversify mathematics education.

Conclusion

The development of children's perception of geometrical concepts is related with the environment where early age children are brought up and gain everlasting and informal knowledge. It is important to combine the knowledge with real life situations and to find motivation for pupil´s work at school. In our article, we wanted to show one of the possible approach, how to use real life object for the creation of the geometrical problems in various levels of difficulty. Most of the common school mathematical tasks are focused on the acquiring of expected knowledge by solving the routine tasks. The motivation and the creativity are very important aspects of education. According to the feedback from the pupils at lower secondary school after solving presented tasks, it was not easy but interesting work.

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ACTA MATHEMATICA 17

STIMULATION OF EXECUTIVE FUNCTION 'SHIFTING' IN TEACHING MATHEMATICS

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ABSTRACT. *Executive functions are mental processes which manage, control and organise human cognition. They represent the fundamental level of mental functioning. The research team from the Faculty of Education, University of Presov is designing a comprehensive program for stimulation of executive functions of pupils aged 9-10 years within the APVV project scheme. Three executive functions: inhibition, working memory and shifting were specified to be stimulated by the program. Mathematical component of the program is focused on stimulating and reinforcing pupil's executive functioning on the background of mathematical curriculum. This paper provides an outline for designing tasks aimed at stimulating the executive function of shifting. Shifting is the ability to switch fluently between multiple sets of cognitive operations. The authors also propose the guidelines for assessing and interpreting the performance of pupils in such tasks.*

KEY WORDS: *Executive Functions, Primary Mathematics, Cognition, Teaching*

CLASSIFICATION: *C32, Q32, U92*

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Introduction

Child's ability to learn as well as the ways to enhance child's cognitive functioning and devising efficient tools to develop child's thinking have become the area of interest, in recent years, not only for psychologists but also for teachers. In a constructivist approach to mathematics different mathematical concepts and their representational models are progressively developed in the minds of every pupil [4]. Such mathematics should be understood as a process of constructing knowledge; a specific cognitive activity aimed at developing thinking. The traditional scope of school mathematics offers many ideas for designing activities which build upon knowledge construction and thus develop cognitive functions and processes.

Developing cognitive functions has a considerable influence on the ability to learn. The above mental functions are referred to as executive functions. Specific level of executive functioning is a prerequisite for the ability to learn. Executive functions are mental tools involved in processing the contents of a school subject [5].

Findings in neuropsychology and behavioural sciences, researches into artificial intelligence and information processing as well as the theory of processing information indicate that it is useful to examine the processes of learning from the cognitivist perspective, the domain in which *executive functions* play a key role [5]. Executive functions represent the fundamental level of mental functioning in that they control and organise the interaction of cognitive functions [7]. Executive functions, in relation to cognitive functions, should be perceived as mental processes which manage cognitive functions.

The research team from the Faculty of Education of the University of Presov is trying to design a battery of tasks aimed at stimulating a selected range of executive functions in pupils aged 9-10 years as one of the aims within the project of APVV scheme. The battery

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of tasks is designed within the primary math curriculum. The individual tasks are gradually elaborated and structured so as to develop a comprehensive stimulation program. Stimulation program in mathematics is meant to be a set of tasks, stimuli and activities designed to stimulate and reinforce mental processes which involve one or more executive functions on the background of mathematical curriculum.

Based on the findings of this project the following three executive functions were specified for the stimulation program. They are: inhibition (attention control), working memory and cognitive flexibility (shifting). The set of tasks developed to stimulate working memory was presented in the article [6].

This paper deals with the design for a stimulation program aimed at stimulating the executive function of *shifting*. The shifting is the ability to switch fluently between multiple sets of cognitive operations. This function is necessary for flexible thinking and action, since it enables to change flexibly the perception and execution of a number of divergent mental operations when solving a problem. Opposite to shifting is so called perseverance, i.e. a rigid adherence to the established way of reacting irrespective of any alternation in the requirements resulting from the situation or task. The tasks on shifting are based on switching between at least two different solutions performed practically at the same time on an identical set of elements or they can include switching between the different criteria for a solution. Since the difficulty of "switching" is positively associated with the reaction time, it is appropriate to include the speed rate when increasing the difficulty of such system.

Several examples of shifting tasks used for testing pupils' executive functioning are described in the literature. For example, St Clair-Thompson & Gathercole [8] introduced *the plus–minus task*. It consists of three lists of 30 two-digit numbers pre-randomized without replacement. On the first list the participants were instructed to add 3 to each number. They were told to complete as many as possible within 2 minutes. Within the same time limit, on the second list, the participants were instructed to subtract 3 from each number, and on the third list the participants were required to alternate between adding and subtracting 3 from the numbers. The cost of shifting was then calculated as the difference between the number of correct answers given in the alternating list and the average of those in the addition and subtraction lists within the given time periods.

Another example of shifting task, introduced in [8], is *the local–global task*. It is a sets of figures in which the lines of a global figure (for example, a triangle) are composed of smaller local figures (for example, squares). On one list, participants were instructed to record the number of lines in the global figure (that is, one for a circle, two for an X, three for a triangle, and four for a square). They were instructed to complete as many as possible within 2 minutes. Within the same time limit, on the second list participants were instructed to record the number of lines in the local figure, and on the third list participants were required to alternate between recording the number of lines in the local figure and recording the number in the global figure. The cost of shifting was then calculated as the difference between the number of correct answers given in the alternating list and the average of those in the local and global lists within the given time periods.

The *Wisconsin Card Sorting Test* (WCST) was designed by D. A. Grant and E. A. Berg [2]. The new manual of WCST is described in [3]. It operates with a series of cards identified by three dimensions: colour, shape and number. Four key cards are placed in front of the child, each with a different shape (triangle, star, cross, or circle), different numbers of shapes (one, two, three, or four), and different colours (red, green, yellow, or blue). The child is asked to pick up the first card from a pile of cards and match it to one of the key cards by arbitrary criteria (colour, shape or number). If the child matches the card by the correct sorting criteria the child should continue sorting subsequent cards by the same dimension. If the matching dimension was incorrect, the child should match the next card by a different dimension, in an attempt to identify the correct one. When the child has maintained the correct sorting dimension for 10 consecutive trials, the matching criteria is changed without explicitly telling the child. It is the child's task to use the feedback to determine that a previous matching criteria that was correct is now incorrect and that a different matching criteria needs to be used. This procedure continues until the child completes six category changes or runs out of cards (total = 128 trials). Results from the WCST show that the main difficulty for children with lower level of mathematical ability is with inhibiting a learned strategy and switching to a new strategy [1].

Stimulation Program: Mathematical Traffic Lights

Following is a proposal for the stimulation program called Mathematical Traffic Lights inspired by Wisconsin Card Sorting Test [2]. Such stimulation program can be applied within teaching mathematics to automate basic connections in mental arithmetic. It is aimed at developing skills in the range of basic mathematical operations with natural numbers. The emphasis is on stimulating the executive function - shifting. It utilises visual presentation and other possibilities made available by ICT and visual educational technologies especially the dynamics of animation (stimulation program can be prepared as a PowerPoint presentation on an interactive whiteboard, or, if the school is sufficiently equipped, it may be posted on website for pupils using tablet). The stimulation program consists of four modules (M1) - (M4) with their modifications.

(M1) This module is designed for a frontal work with pupils as a math warm-up. The basic scheme is as follows:

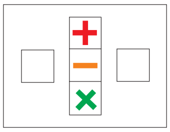

Figure 1

The numbers are inserted into the empty boxes and the tasks are activated by switching a "traffic light". Pupils gradually solve a set of tasks (the number of tasks is arbitrary by the teacher).

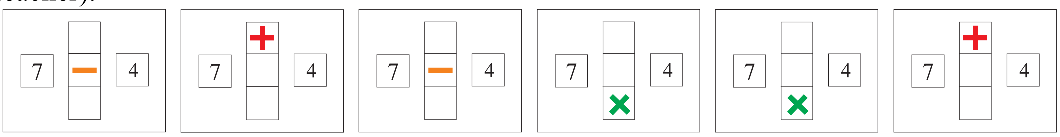

Figure 2

In the next set of tasks the colour of the sign which indicate the operation is changed. In the last set of tasks switching traffic lights is combined with a changed pair of the numbers.

Further modifications of the basic scheme may look like this:

Figure 3

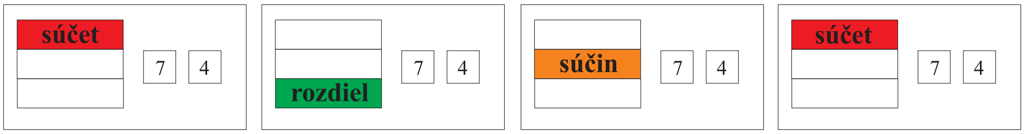

Figure 4

(M2) The module can be used in frontal work with pupils as a math warm-up. The basic scheme is as follows:

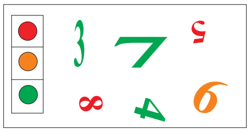

Figure 5

The pupil's task is to name the numbers of the same colour which appear on the traffic light. In the first set of tasks the numbers remain the same only the traffic lights are switched over. In the next set of tasks switching over the traffic lights is combined with changing the numbers. Modifying the above scheme we obtain mathematical exercises aimed at developing basic mathematical operations with natural numbers. The basic scheme is the following:

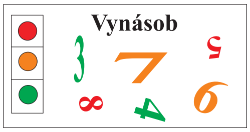

Figure 6

The pupil's task is to multiply the numbers of the same colour that appears on the traffic light. In the first set of tasks the numbers remain the same only the traffic lights are switch over. For example:

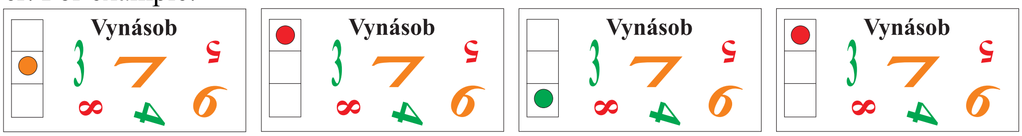

Figure 7

In other tasks the colour of the numbers is switched simultaneously with the change in the traffic light by which we always obtain new pairs of the same colour. Finally, the pupils are presented a set of tasks in which the numbers are changed. Further modifications of this scheme can be obtained if the operation is changed or a geometric figure is included or using two traffic lights while switching between colours and characters which denote the operation and so on. The basic scheme may look like this:

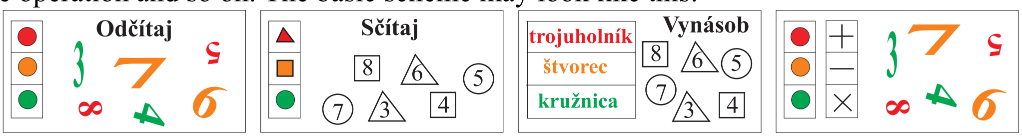

Figure 8

(M3) This module is designed for individual work with pupils. The basic scheme is as follows:

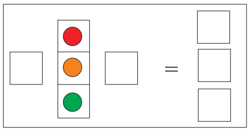

Figure 9

The numbers are inserted into the empty boxes. The task is specified by switching traffic lights, for example:

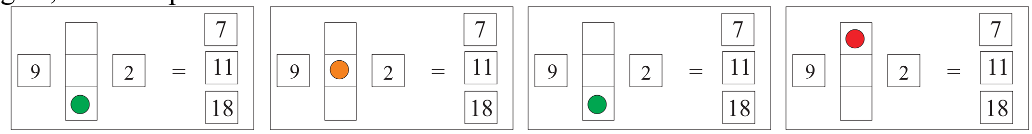

Figure 10

Each colour is assigned to one of the operations $+,-,\times$ (e.g. red $+$, green $\times$ and orange $-$). After the traffic light is on, a pupil selects one of the numbers in the right column after which he is given a feedback whether he had chosen the right or wrong number. The pupils' task is to determine which operation is assigned which colour. The pupil can be informed in advance that each colour represents one operation, but the legend is not disclosed.

The most challenging modification of this scheme is when switching the traffic light is combined with the change of the number at the same time.

Pupil's performance can be interpreted in two ways.

1. The pupil is given thirty tasks in three rounds, ten tasks in each round. After each round the legend is changed and the student is informed of that. The aim is that the pupil identifies the legend and adopts (switches to) a new rule. Pupil's achievement is assessed on the basis of the number of correct answers, whereas only the last five tasks in each round are considered. The first five tasks in each round serve as calibration so that the pupil is left room to detect a rule.
2. An appropriate time limit is set in advance. The task is then presented to pupils. When the administrator sees that the pupil discovered the rule and determined the right solution in, say, four consecutive tasks, the rule is changed. Performance of the pupil is then assessed by the number of discovered rules within the set time limit.

(M4) The basic scheme is as follows:

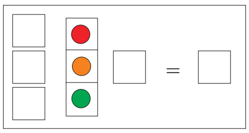

Figure 11

The numbers are inserted into the boxes and the tasks are given by switching over the traffic lights, for example:

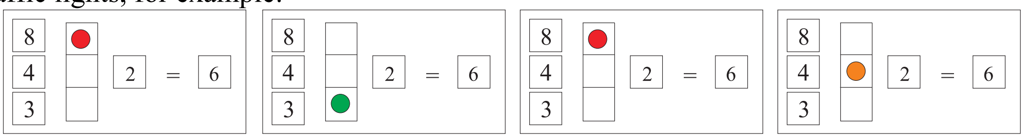

Figure 12

Like in the module (M3), each colour is assigned one sign of operation $+, -, \times$ and the pupil is not familiar with the legend. The pupil selects one of the numbers in the left column and after each choice he learns whether the chosen number was right or wrong. The pupil's task is to discover which colour denotes which operation $+, -, \times$. Performance of pupils can be interpreted by any of the two methods outlined in the module (M3). The more demanding modification of this module can be obtained when switching over the traffic light is combined with the simultaneous change of the number.

Conclusion

The paper presents the proposed set of tasks intended for stimulating the executive function "shifting". The stimulation programs Mathematical Traffic Lights and PIN Code [6] are designed as basic schemes focusing on arithmetic content area. The authors' intention is to adapt both stimulation programs to other content areas of mathematical curriculum and to expand the range of stimulation programs to incorporate other executive functions.

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ACTA MATHEMATICA 17

PUPILS´ SOLUTIONS OF A GEOMETRIC PROBLEM FROM A MATHEMATICAL COMPETITION

LUCIA RUMANOVÁ, JÚLIA ZÁHORSKÁ, DUŠAN VALLO

ABSTRACT. *In this article we list a brief analysis of pupils' solutions of a geometric task from the 63th Mathematical Olympiad category Z9. The task was given under regional competition category to the pupils at lower secondary education. We wondered what solutions pupils used and what mistakes occurred in their solutions. We present some solutions of the pupils with specific examples of their sketches.*

KEY WORDS: *competition, analysis, solutions, geometric problem*

CLASSIFICATION: *D54, G44*

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Introduction

According to the viewpoints of many pedagogues as well as researchers from didactics, in order to achieve better results of pupils, it is not enough only to use suitable textbooks or other teaching materials in the teaching process. Pupils must build not only their knowledge, but also an active approach to learning itself. Therefore, for example, problem solving is currently considered a basis for learning.

In PISA study [1] problem solving is defined as an individual ability to use cognitive processes for solving real interdisciplinary problems, when the path to the solution is not immediately visible and the content of knowledge areas, which are necessary to be applied to the solution, is not obvious at first sight.

In the official Slovak document entitled National Program of Education [2] it is also shown that:

- it is necessary to include problematic tasks throughout studies
- the study of mathematics at secondary schools contributes to the development of key competencies for solving of problems, it means: to apply appropriate methods to problem solving which are based on analytical and critical or creative thinking; to be open for capturing and exploiting solving problems with different and innovative practices; to formulate arguments and proofs for defending their results.

As the author claims, in [3] understanding in mathematics, when dealing with also non-mathematical problems and tasks, there is apparently the need for application of mathematical knowledge acquired in non-standard situations. The effort of teachers should encourage students to solve problems in different ways, in regard to their knowledge, skills and acquired mathematical tools.

In this article we list a brief analysis of pupils' solutions of a geometric task from the 63th Mathematical Olympiad category Z9 (the pupils at lower secondary education).

The selected problems from the Mathematical Olympiad and the pupils´ solutions

We have chosen a suitable task of geometry, which was included within the 63th Mathematical Olympiad 2013/2014 as the task of regional competition of category Z9. We wondered what solutions the pupils used and what mistakes occurred in their solutions.

The geometric task: *Within an equilateral triangle ABC there is inscribed an equilateral triangle DEF. Its vertices D, E, F lie on sides AB, BC, AC and the sides of triangle DEF are perpendicular to the sides of triangle ABC (as shown in Figure 1). Also, segment DG is the median of triangle DEF and point H is the intersection of DG, BC. Determine the ratio of triangle HGC to DBE.* [4]

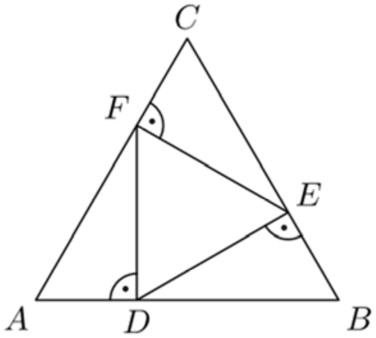

Figure 1: The geometric task

These examples are the solutions of three pupils and the task was solved correctly by 9 of 30 pupils.

<u>The 1st pupil´s solution:</u> In most cases the pupils confirmed that triangle *DEF* is also equilateral, when they added the angles of triangle *ABC*, they found out that the median and height from point *D* of triangle *DEF* are identical. Then they found out that segments AC and DH are parallel (as shown in Figure 2) so $|GH| = \frac{1}{2}|FC| = \frac{y}{2}$, triangles *ADF, BED, CFE* are equal, then$|DB| = |CE|$ and $|DB| = |CE| = |AF| = 2.|HE|$.

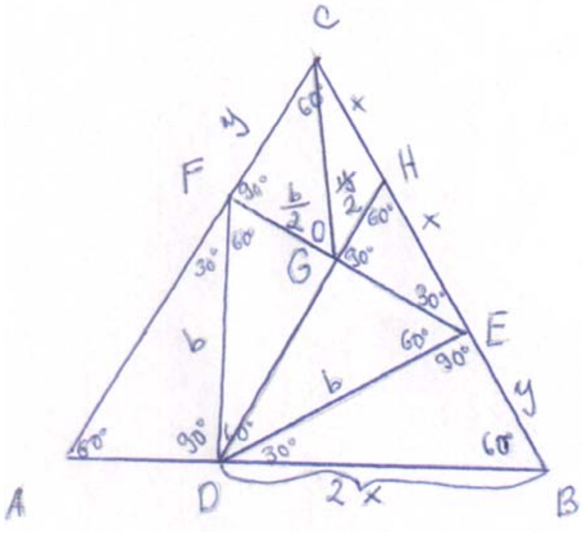

Figure 2: A sketch of a pupil´s solution

They obtained relations for the areas of triangles *FEC, HGE*, *FGC* and *GHC*. Whereas the areas of triangles *CFE* and *BED* are the same, they just put in ratio the areas of triangles *HGC* and *DBE*, which is 1:4.

The 2nd pupil´s solution: The pupil used the following pictures for solving the task (see Figure 3).

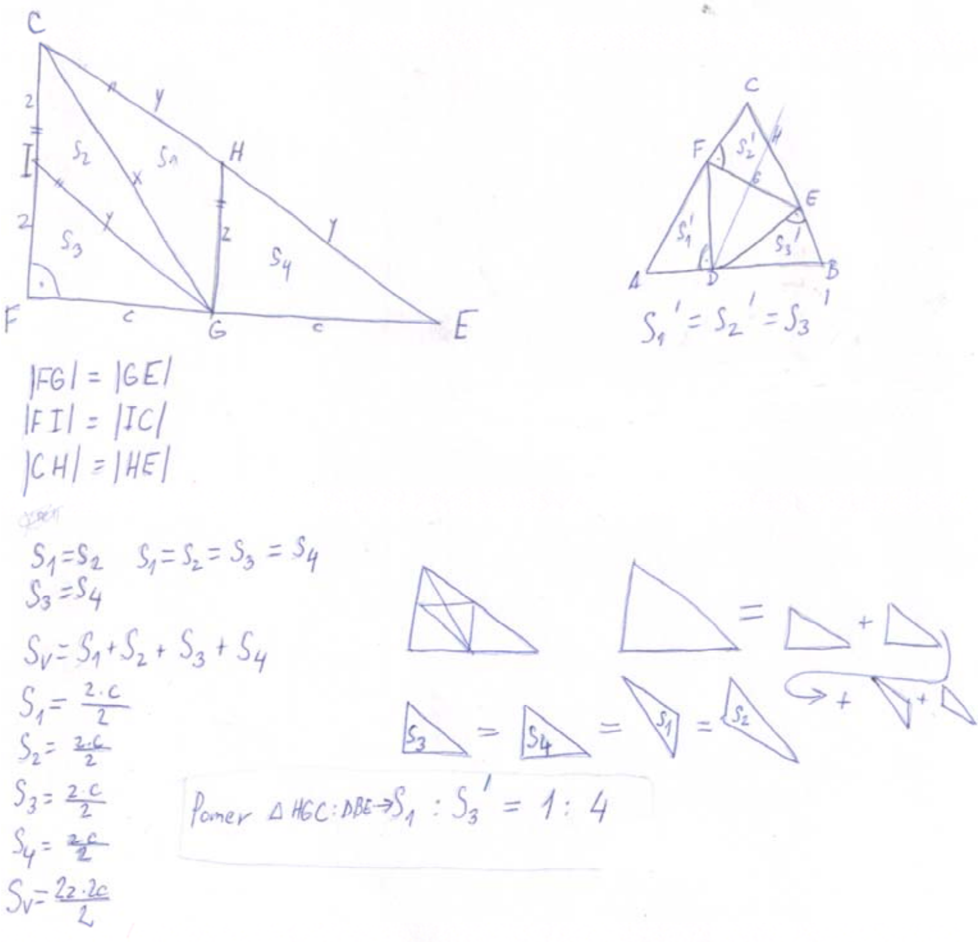

Figure 3: The 2nd pupil´s solution

The 3rd pupil´s solution: Other pupil´s solution contained three pictures. As we can see in Figure 4, beginning of the task solution is identical to the solution of the first pupil in our article.

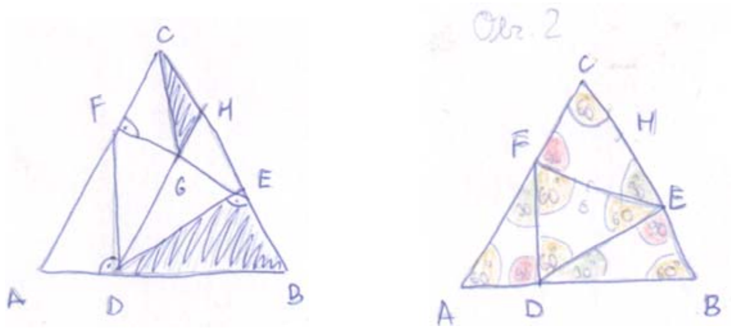

Figure 4: The 3rd pupil solution

Then in Figure 5 there is a detail of triangles *FEC* and *HGC*, so from the existing relations it is valid that: $|HX| = |HG|, |FG| = |GE|$ and the areas of triangles *GEH, HXC* are the same. So the area of quadrilateral *FGXC* is the same as the area of the right triangle *FEC*. The area of triangle *HCX* is equal to a half of the area of triangle *GXC*. Whereas the area of triangle is *GXC* is again a half of the area of the right triangle *FEC*, so the area of the triangle is equal to $\frac{1}{4}$ of its area. Therefore, the ratio of the triangles is $|HGC|: |DBE| = 1: 4$.

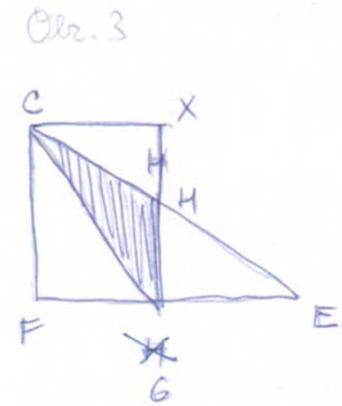

Figure 5: A detail of the triangle

Mistakes in the pupils´ solutions

In this part of the article there is a list of the most frequent mistakes that we have seen in the pupils´ solutions. So incorrect solutions were mainly:

- a graphical solution of the problem – complementing the known facts into pictures (see in Figure 6) followed by the result of the solution,

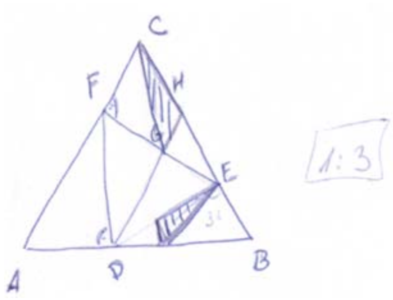

Figure 6: One graphical solution of a pupil

- an incorrect approach in the first step of the solution, for example: segment *DG* is the median of triangle ABC which means $|BE|:|EC| = 1:2$,
- from incorrect facts some got the correct result, for example: in the solution a pupil used the property that the median divided triangle into two equal triangles in the ratio 2:1,
- pupils found that triangles *BED* and *CFE* are equal or determined the size of some angles and then considered the identity of the median and height, but other considerations were not correct,
- searching for the area ratio of trapezoid *CFGH* to triangle *HGE* (3:1), but then the faulty conclusion followed and the pupils also used a fictional length,
- they determined that the right triangles are identical, then they completed the picture, however, without explaining other facts,
- measuring the lengths of the sides and heights of the triangles, they followed with other calculations or finding out of the fact that the right triangles are identical,
- confirmation that triangle *FHC* is an equilateral triangle, then finding out that $|CH| = \frac{1}{3}|CB|$ and the next steps of the solution were wrong,
- the solutions often contain fictional dimensions in the picture and so the next steps were incorrect.

Conclusion

We can observe the fact that when solving the given task, the pupils (who have more than average mathematical skills because they progressed to the mathematical competition) can solve a more demanding geometric task at different levels. The pupils have proven their individual skills in their solutions, individual solving strategies, their registrations or justification.

These results just like the study by [5] confirm the fact that pupils of certain age, assuming the appropriate level of solving and mathematical skills and knowledge, build and develop skills to solve mathematical problems by finding and choosing their own solution strategies.

Therefore, in supporting the work with talented pupils we can see the way to develop pupil´s individuality.

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ACTA MATHEMATICA 17

MOTIVATION TO GEOMETRY AT HIGH SCHOOL OF VISUAL ARTS

EDITA SMIEŠKOVÁ[1], EVA BARCÍKOVÁ

ABSTRACT. *In the article we deal with the problem of lack of pupils' motivation to geometry in high school of visual arts. Our theoretical framework is van Hiele model of Geometry Thought. We prepared one lesson from mathematics based on activities fitting on the first and second level of the van Hiele model. One of the aim was to show pupils a cross curricular relation between mathematics and visual arts.*

KEY WORDS: *teaching geometry, motivation, visual arts*

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Introduction

In our opinion, the most beautiful part of teaching mathematics is geometry. Geometry includes many interesting problems and theories and is still open to new approaches. The history of geometry comes from the time of creating the mankind where it was a part of a real life and culture. In the past, the geometry was the component of architecture in many various forms. Mathematics as a teaching subject belongs to less popular subjects for many different reasons. One reason could be that not so many students are successful in mathematics. According to [1] students' perception of success in mathematics has a great effect on students' motivation attitudes. And we think that maybe the teaching of geometry in a more interesting way could be an element of motivation for learning of students, because geometry has an influence on our aesthetical and visual perception of the world. In the present, the real life is often used in the connection of mathematics, however, students do not find the connection between mathematics and their experiences.

Why is important to teach geometry?

The world around us is a visual. Since we live in 3D space, it means, that it is necessary to know interpret visual information. See the visual arts, architecture and many other cultural artifacts from an aesthetical point of view, includes geometrical principals: symmetry, perspective, an orientation in space and so on. The teaching subject geometry offers a rich way of developing the visual abilities of students. "*No mathematical subject is more relevant than geometry. It lies at the heart of physics, chemistry, biology, geology, and geography, art and architecture. It is also lies at the heart of mathematics, though through much of the 20th century the centrality of geometry was obscured by fashionable abstraction*". [2]

In the article, we will talk about 15 and 16 years old pupils of the high school of visual arts. In our opinion geometry, especially spatial imagination is a necessary ability for them and for their future work.

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How to teach geometry?

According to [3] and [4], if the activities in geometry are frustrating and not interesting, then pupils might not be motivated to learn what the teacher is trying to teach them. On the other hand, if the activities are too easy, they might not attract pupils' attention to the topic. So they might fail to generate a sense of success. The tasks in instruction should contain respectable challenges that pupils could achieve. With appropriate students' ability to achieve geometry tasks dealt many educators and researchers. One of them was Dina van Hiele-Geldof and Pierre van Hiele. We placed our work into the theoretical framework based on their Van Hiele model of Geometric Thought. It is a theory in mathematics education, which describes the way how students reason about shapes and other geometric ideas. Five discrete hierarchical levels of thinking were described in this model [5]:
Level 0 (Visualization): Students recognize the figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. At this level, students make decisions based on perception, not reasoning.
Level 1 (Analysis): Students see figures as sets of properties. They can recognize and name properties of geometric figures, but they do not see relationships among these properties. When describing an object, a student operating at this level might list all the properties he/she knows, but may not discern which properties are necessary and which are sufficient to describe the object.
Level 2 (Abstraction): Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. The role and significance of formal deduction are, however, not understood.
Level 3 (Deduction): Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. At this level, students should be able to construct proofs such as those typically found in a high school geometry class.
Level 4 (Rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems. Students at this level can understand the use of indirect proof and proof by contrapositive, and can understand non-Euclidean systems.

Every of these levels has its own linguistic symbols and its own systems of relations connecting this symbol. It is important for the teacher to know on which level their students are to give them suitable instructions, which there are able to understand. Although the activities on lesson should fit their level.

Students in our study group have low levels of geometrical thinking. We could say that the level of thinking of our study group has been just first or second. As we write below they were not interested in math, neither geometry. For this type of students is especially desirable to choose the appropriate approach of teaching geometry. For these levels are proper also manipulating activities and didactical plays. Activities fitting the first and the second levels could be for instance:

- to manipulate with geometric shapes,
- to identify a shape or geometric relations in simply physical objects in the classroom, the home, photographs, and other places,
- to compose and decompose shapes,
- to solve tasks where properties of figures and interrelationships are important

Should be the teaching mathematics at the high school of visual arts effective?

In the Private high school of visual arts in Nitra is mathematics teach once a week in the first and second year of the study in school year 2013/2014, what is 33 hours per school year. Should be so lessons effective? If we want to teach everything from the curriculum, it will not be possible to do so. Moreover, the textbooks are so difficult for this type of school and there does not exist textbook of mathematics for high schools of visual arts. For this reason is sometimes really hard for teachers to prepare for the lesson. The age of pupils in our study group is between 15 and 16 years. It is an age when pupils are not interested in studying generally and they are searching for answers of the sense of life.

We prepared one lesson of geometry, based on activating teaching strategies. One of our main aim was to show pupils the geometry in a real life. The next aim was to make a lesson for each student, also for students who are not interested in mathematics and they are not so successful and give them a chance to be an active. We prepare for them activities fitting the first and the second level of the van Hiele model. Through these activities pupils have the opportunity to learn, explore geometry by their own way and tempo. The thought was also to show students a cross curricular relation between mathematics and visual arts.

Geometric figure in the plane

For the lesson of mathematics pupils needed their own cameras. The teaching lesson began with an introduction talk about geometry and our surroundings. The first task was to take a picture of any geometrical figure in their class. Then the second task was to take a picture of any geometrical figure in their school. The last picture pupils had to take outside of the school building. After coming back to the classroom, pupils had to figure out their own tasks inspired by their pictures, if they did not know to figure out tasks they should compose a poem or a short story.

Students after first pictures were motivated to work and they took more than three pictures. Every pupil in the class made this task and we could see that this activity is likely for them.

Now, for instance, we offer some tasks of pupils':

1. The area of the circle with radius *11 cm* is *x* cm^2. The circle is divided in two not the same parts. Red part of the circle has the area 421 cm^2. How many percent of a circle's area is a rose part of the circle?

Figure 1

2. The sum of the same three circles' areas is the same as the area of the rectangle. Radius of one circle is 3cm. Sides of the rectangle are *a* = ? and *b* = 7 cm. What is scale of side *a*?

Figure 2

3. Peter weights *70* kilograms and one his step measure *80 cm*. In the one lazy Friday Peter was walking from the school, he was joking by the jump of the canals. Each canal he jumped on one his step. During the way he jumped 11 canals.

a) What is the sum of radiuses all jumped canals?

b) What is the sum of canals' hoods weight, if the one weight is 14 times less as Peter's weight?

Figure 3

4. Pupils also created a short poem:

It keeps us on the ground,
it keeps us in the driving,
whether it's to the Skoda
or Mazda.
What is this?
(Wheel)

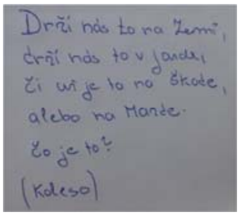

Figure 4

Conclusion

Geometry found its beginning due to a utilitarian need to understand and predict phenomena in the natural world. Although the world and society changed, geometry still gives us a tool to understand our environment. That is one of the reasons, why we found geometry to be an important part of school mathematics. Especially, we see the needs of geometry in high school of visual arts, where students deal with geometrical principals in architecture or linear perspective. However, in this type of school is low dotation of math classes and also a problem with lack of pupils' interest. Due to these problems we tried to find the way to motivate students to learn geometry. We prepared one lesson of geometry with activities fitting pupils' levels of geometry thinking. During this lesson, students have the opportunity to observe the geometrical shapes in their environment. Every pupil, were not interested in math, participated in these activities and they enjoyed it. So we consider our aim to be satisfied. Our plans to the future is to prepare next lessons with the similar aim to active pupils in the teaching of mathematics, because we think that one lesson of the week must be also effective and must bring some mathematical knowledge to pupils of the high school of visual arts.

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ACTA MATHEMATICA 17

FOURIER TRANSFORM AND ITS APPLICATION

DARINA STACHOVÁ

ABSTRACT. *By means of Fourier series can be described various examples of wave motion, such as the sound, or wave of the earthquake. It can be used in many research or work, such as the data analysis after an earthquake or digitizing music. Generalization of Fourier series, which allows for some applications more appropriate expression is the Fourier integral. Fourier transform based on a Fourier integral in the complex form. Fourier transform is an important tool in a number of scientific fields. Its advantages, disadvantages and subtleties have been examined many times by dozens of mathematicians, physicists and engineers. In this contribution we try to summarize important aspects of this transform and discuss variety of its uses in contemporary science with emphasis on demonstrating connections to dynamic interactions in the vehicle-roadway system.*

KEY WORDS: *Fourier transform, time series, frequency representation.*

CLASSIFICATION: *I55, I85, M55*

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1 Introduction

Empirical measurements in various domains – economical, technical, or other – are often turned into time series. Based on this it is possible to perform analysis, which in turn allows us to better understand the dynamics of the factors involved. To this end, we use the Fourier transform. Being discovered at the turn of the 19th century, the theory of the Fourier transform is currently used in signal processing such as in image sharpening, noise filtering, etc. For us the relevant application is in the theory of dynamic interaction in the vehicle-roadway system.

The basis of the Fourier transform is the so-called Fourier mapping, i.e., the transformation of one function to another, from properties of which we can obtain information about the original function. Fourier transform expresses a time-dependent signal using harmonic signals, i.e., the sine and cosine functions, in general functions of complex exponentials. It is used to transform signals from the time domain to the frequency domain. A signal can be continuous or discrete. See Figure 1 for a depiction of the correspondence between the time-based and frequency-based representation.

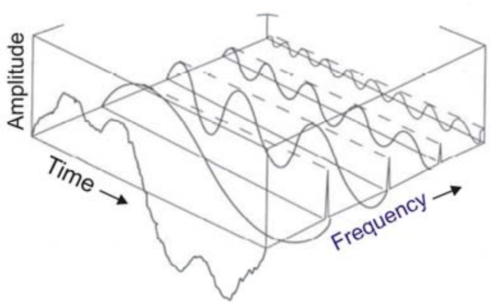

Figure 1: Amplitude frequency diagram

2 Fourier integral

A generalization of the Fourier series that permits in some applications a more appropriate expression of a non-periodical function defined almost everywhere in R is the Fourier integral.

Theorem 1: Let $f: R\to R$ be a function that

a) is piece-wise continuous on R along with its derivative f',

b) is absolutely integrable on R, i.e., $\int_{-\infty}^{\infty}\left|f(t)\right|\,\mathrm{d}t$ converges.

Then all $t\in R$ satisfy

$$\tilde{f}(t)=\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\int_{-\infty}^{\infty} f(s)\cos\omega(t-s)\mathrm{d}s\,,\ \text{where}\ \tilde{f}(t)=\frac{1}{2}\left[\lim_{s\to t+}f(s)+\lim_{s\to t-}f(s)\right]. \tag{1}$$

Note 1: From the claim of Theorem 1, it follows that the values of the double integral on the right-hand side of (Eq. 1) is equal to $f(t)$ for each $t\in R$ in which f is continuous and is equal to the arithmetic mean of the left- and right- limits of this function in each point of discontinuity, provided the conditions a) and b) of this claim hold.

Note 2: Using the fact that $\forall\, t\in R$, $\forall\, s\in R$ and $\forall\,\omega\in\langle 0;\infty)$ we have $\cos\omega(t-s) =$ $=\cos\omega t\cos\omega s+\sin\omega t\sin\omega s$, we can rewrite (1) in the form

$$\tilde{f}(t)=\int_0^\infty\left[a(\omega)\cos\omega t+b(\omega)\sin\omega t\right]\mathrm{d}\omega\,, \tag{2}$$

where $a(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(s)\cos\omega s\,\mathrm{d}s$, $b(\omega)=\dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty}f(s)\sin\omega s\,\mathrm{d}s$, $\omega\in\langle 0;\infty)$. (3)

Definition 1: The right-hand side of (Eq. 1) is called the *double Fourier integral* of $f: R\to R$. The right-hand side of (Eq. 2) is the *single Fourier integral* of f.

Note 3: It is not difficult to see that the single Fourier integral (Eq. 2) is a generalization of the double Fourier integral and the functions a: $\langle 0;\infty)\to R$, b: $\langle 0;\infty)\to R$ defined by (Eq. 3) are a generalization of the standard Fourier coefficients of a periodic function. It is clear that if $f: R\to R$ is an even function, then $b(\omega)=0$, $a(\omega)=\frac{2}{\pi}\int_0^\infty f(s)\cos\omega s\,\mathrm{d}s$ and $\tilde{f}(t)=\int_0^\infty a(\omega)\cos\omega t\,\mathrm{d}\omega$ $=\frac{2}{\pi}\int_0^\infty \mathrm{d}\omega\int_0^\infty f(s)\cos\omega s\cos\omega t\,\mathrm{d}s$. Similarly, if f is odd, we have: $a(\omega)=0$, $b(\omega)=\frac{2}{\pi}\int_0^\infty f(s)\sin\omega s\,\mathrm{d}s$ and we have $\tilde{f}(t)=\int_0^\infty b(\omega)\sin\omega t\,\mathrm{d}\omega$ $=\frac{2}{\pi}\int_0^\infty d\omega\int_0^\infty f(s)\sin\omega s\sin\omega t\,\mathrm{d}s$.

Note 4: Using the well-known Euler's formula for exponential and goniometric functions: $\cos\omega t=\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{e}^{-\mathrm{i}\omega t}\right)$, $\sin\omega t=\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{-\mathrm{i}\omega t}\right)$ in the single Fourier integral (Eq. 2), we obtain $\forall\, t\in R$: $\tilde{f}(t)=\displaystyle\int_0^\infty\left[\frac{a(\omega)-\mathrm{i}b(\omega)}{2}\mathrm{e}^{\mathrm{i}\omega t}+\frac{a(\omega)+\mathrm{i}b(\omega)}{2}\mathrm{e}^{-\mathrm{i}\omega t}\right]\mathrm{d}\omega$.

By letting $\frac{a(\omega)-\mathrm{i}b(\omega)}{2}=c(\omega)$, $\frac{a(\omega)+\mathrm{i}b(\omega)}{2}=c(-\omega)=\overline{c(\omega)}$ for $\omega\in\langle 0;\infty)$, we have for all $t\in R$:

$$\tilde{f}(t)=\int_{-\infty}^{\infty}c(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega\,, \tag{4}$$

where $c(\omega)=\frac{1}{2}\left[a(\omega)-\mathrm{i}b(\omega)\right]=\frac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}f(s)\mathrm{e}^{-\mathrm{i}\omega s}\mathrm{d}s$ for all $\omega\in R$. (5)

Definition 2: The right-hand side of (4), where $c: R\to C$ is defined by (5) is called the *Fourier integral* of $f: R\to R$ in a *complex form*.

3 Fourier transform

Definition 3: Let f: $R \to R$ along with its derivative f' be piece-wise continuous on R, and let f be absolutely integrable on R. Then we call f the *source of Fourier transform*. Let the set of such functions f: $R \to R$ be denoted by D_F. Then the function F: $\mathrm{i}\omega \to F(f(t))$,

where $F(f(t)) = F(\mathrm{i}\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$, (6)

where $\omega \in (-\infty;\infty)$ is called the *Fourier image* of f and the mapping F from the set of functions D_F defined by (Eq. 6) is called the *forward Fourier transform*.

Theorem 2: If $f \in D_F$, then there exists a Fourier image $F(i\omega)$ of f defined by (Eq.6), which satisfies $\tilde{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$, (7)

where $\tilde{f}(t) = \frac{1}{2}\left[\lim_{s \to t+} f(s) + \lim_{s \to t-} f(s)\right]$ for all $t \in R$.

Definition 4: The mapping $F^{-1}(D_F)$ defined by (7) is called the *inverse Fourier transform*, i.e., $F^{-1}(F(\mathrm{i}\omega)) = \tilde{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$, $t \in R$.

Note 5: The Fourier image $F(\mathrm{i}\omega)$ is in the technical literature often called the *spectral characteristic* of f. Its magnitude $F(\omega) = |F(\mathrm{i}\omega)|$ is the *amplitude characteristic* of f, function $\alpha(\omega) = -\operatorname{Arg} F(\mathrm{i}\omega)$, $\omega \in (-\infty;\infty)$ is called the *phase characteristic* of f and the function $P(\omega) = |F(\mathrm{i}\omega)|^2$ *power characteristic* (*power spectrum*) of f. Hence, for all $\omega \in R$

$F(i\omega) = F(\omega)\,\mathrm{e}^{-\mathrm{i}\alpha(\omega)} = A(\omega) - \mathrm{i}B(\omega)$, where $A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t$, $B(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t$

.

From this it follows that $F(\omega) = \sqrt{A^2(\omega) + B^2(\omega)}$, $\alpha(\omega) = \arctan[B(\omega)/A(\omega)]$, which means that the amplitude function $F(\omega)$ is an even function and the phase function $\alpha(\omega)$ is an odd function of the independent variable (frequency) ω.

4 Use of Fourier transform in solving representative problems

Recall that according to the Euler's formula we can write $\mathrm{e}^{\mathrm{i}\omega t} = \cos\omega t + \mathrm{i}\sin\omega t$.

Example 1: Find the Fourier image of the function $f(t)$: $R \to R$, $f(t) = \mathrm{e}^{-a|t|}$, $a \in R^+$.

Solution: From equations (Eq. 4) and (Eq. 5) it follows that

$$c(\omega) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-a|s|}\,\mathrm{e}^{-\mathrm{i}\omega s}\mathrm{d}s = \tfrac{1}{2\pi}\left(\int_{-\infty}^{0}\mathrm{e}^{(a-\mathrm{i}\omega)s}\mathrm{d}s + \int_{0}^{\infty}\mathrm{e}^{-(a+\mathrm{i}\omega)s}\mathrm{d}s\right) = \tfrac{1}{2\pi}\left(\left[\frac{\mathrm{e}^{(a-\mathrm{i}\omega)s}}{a-\mathrm{i}\omega}\right]_{-\infty}^{0} - \left[\frac{\mathrm{e}^{-(a+\mathrm{i}\omega)s}}{a+\mathrm{i}\omega}\right]_{0}^{\infty}\right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{a-\mathrm{i}\omega} + \frac{1}{a+\mathrm{i}\omega}\right) = \frac{1}{\pi}\frac{1}{a+\omega^2}, \text{ i.e., } \tilde{f}(t) = f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{a+\omega^2}\mathrm{d}\omega.$$

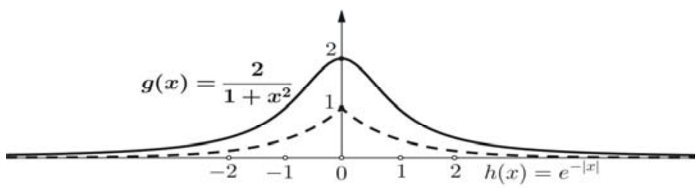

Figure 2: Comparing a graph of a function with its Fourier image

Therefore the Fourier image of $\mathrm{e}^{-a|t|}$ is $F(\mathrm{e}^{-a|t|}) = \frac{1}{a}\cdot\frac{2}{1+\left(\frac{\omega}{a}\right)^2} = \frac{2}{a^2+\omega^2}$.

Fourier transform has a wide variety of uses; we have already shown some of them for illustration. Fourier transform is also used to solve differential equations. The key idea is that the Fourier transform transforms the operation of taking derivatives into multiplication of the image by the independent variable. If we perform the Fourier transform using all independent variables we obtain as image a solution of the equation with no derivatives. When we solve it, it suffices to find the Fourier preimage which usually is the most difficult part. Unfortunately, it can also happen that the solution has no preimage. Then this method does not work. However, we may perform the Fourier transform using only some independent variables. This yields a differential equation with fewer variables and with parameters, which might be easier to solve than the original equation; nonetheless the ultimate difficulty may still be in finding the preimage.

Example 2: Using the Fourier transform find a solution of the differential equation satisfying the following conditions:

a) $y'(t) + k\,y(t) = a\,e^{-|t|}$, where $k \in R^+ - \{1\}$, $b \in R$, $t \in R$, $\lim\limits_{t\to-\infty} y(t) = \lim\limits_{t\to\infty} y(t) = 0$,

b) $y''(t) + 3y'(t) + 2\,y(t) = e^{-|t|}$, $t \in R$, $\lim\limits_{t\to-\infty} y(t) = \lim\limits_{t\to\infty} y(t) = 0$.

Note: Fourier transform can also be used for solving ordinary linear differential equations with constant coefficients assuming that the solution of such equation along with its derivatives of order up to the order of the equation has properties from Definition 3.

Solution: Let y, y', $y'' \in D_F$ and write $F(y(t)) = Y(i\omega)$. Then $F(y'(t)) = i\omega Y(i\omega)$, $F(y''(t)) = -\omega^2 Y(i\omega)$.

a) Since $F(a\,e^{-|t|}) = a\dfrac{2}{1+\omega^2}$, we have $y(t) = \dfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} Y(i\omega)e^{i\omega t}dt = \dfrac{a}{\pi}\int\limits_{-\infty}^{\infty}\dfrac{1}{1+\omega^2}e^{i\omega t}dt$.

Thus $y(t) = \dfrac{ae^t}{k+1}$ for $t \in (-\infty;0)$, $y(t) = a\left(\dfrac{e^{-t}}{k-1} - 2\dfrac{e^{-kt}}{k^2-1}\right)$ for $t \in \langle 0;\infty)$.

b) Since $F(e^{-|t|}) = \dfrac{2}{1+\omega^2}$, it follows that $y(t) = \dfrac{1}{2\pi}\int\limits_{-\infty}^{\infty} Y(i\omega)e^{i\omega t}dt = \dfrac{1}{\pi}\int\limits_{-\infty}^{\infty}\dfrac{1}{1+\omega^2}e^{i\omega t}dt$.

Hence $y(t) = \dfrac{1}{6}e^t$ for $t \in (-\infty;0)$, $y(0) = \dfrac{1}{6}$, and $y(t) = \dfrac{2t-1}{2}e^{-t} + \dfrac{2}{3}e^{-2t}$ for $t \in (0;\infty)$.

Example 3: Some situations require specifying the problem using a diagram. Here the subject of analysis is the so-called *quarter model* of a vehicle shown in Figure 3. This computational model represents one half of one axle of a vehicle. Unevenness of the road surface is the main source of kinematic excitation of the vehicle. The vehicle's response to this excitation can be found numerically in both the time and frequency domain. In the time domain we are mainly interested in time evolution of contact forces and in the frequency domain in the power spectral densities of power forces in relation to the power spectral densities of the unevenness of the road [1].

As an example, we use numerical characteristics of the vehicle Tatra model T148.

Weight parameters of the model: $m_1 = 2\,930$ kg $\quad m_2 = 455$ kg

Rigidity constants of the coupling: $k_1 = 143\,716{,}5\ \mathrm{N\cdot m^{-1}}$ $\quad k_2 = 1\,275\,300{,}0\ \mathrm{N\cdot m^{-1}}$

Damping coefficients: $b_1 = 9\,614{,}0\ \mathrm{kg \cdot s^{-1}}$

Note: Inherent part of the process of solving the problem is the formulation of simplified models of the vehicle, their mathematical description, and determination of the vehicle's response in the time domain. Computational models of vehicles can have varied complexity depending on the nature of problem to be solved. Oftentimes the so-called quarter- of half models are used; these models model motion and effects of a quarter or half of the vehicle. Nowadays, however, it is not uncommon to use spatial models of vehicles.

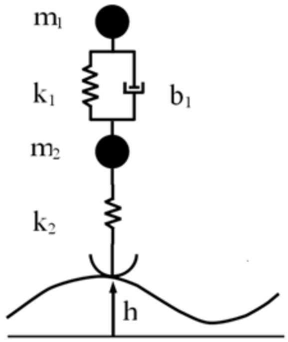

Figure 3: Quarter model of a vehicle

The law of conservation of mechanical energy is a special case of the conservation of energy law, which applies to all types of energy. In the case of dissipative forces such as frictional forces, part of the mechanical energy is converted to heat, but the total amount of energy remains the same.

Solution: Applying a general procedure [2] to the model from Figure 3, we obtain equations of motion of the modeled vehicle. With that we also obtain expressions describing interaction forces at the point of contact of the vehicle's axle with the road surface.

$$r_1''(t)m_1 = \{-k_1[r_1(t) - r_2(t)] - b_1[r_1'(t) - r_2'(t)]\}$$
$$r_2''(t)m_2 = \{+k_1[r_1(t) - r_2(t)] - k_2[r_2(t) - h(t)] + b_1[r_1'(t) - r_2'(t)] - b_2[r_2'(t) - h'(t)]\} \tag{8}$$

Using the principle of equal action and reaction, we derive the following:

$$F(t) = -F_{RV}(t) = -G_2 + k_2[r_2(t) - h(t)] + b_2[r_2'(t) - h'(t)] = F_{st} + F_{dyn}(t),$$

i.e., $F_{st} = -G_2$ and $F_{dyn}(t) = k_2[r_2(t) - h(t)] + b_2[r_2'(t) - h'(t)]$. (9)

We rearrange the equations (8) as follows:

$$m_1 r_1''(t) + b_1 r_1'(t) - b_1 r_2'(t) + k_1 r_1(t) - k_1 r_2(t) = 0$$
$$m_2 r_2''(t) - b_1 r_1'(t) + b_1 r_2'(t) + b_2 r_2'(t) - b_2 h'(t) - k_1 r_1(t) + k_1 r_2(t) + k_2 r_2(t) - k_2 h(t) = 0$$
$$F_{dyn}(t) = b_2 r_2'(t) - b_2 h'(t) + k_2 r_2(t) - k_2 h(t). \tag{10}$$

Function $f(t)$ and its time derivative will be then transformed in this way: $a\,f(t)$ to $a\,F(\omega)$, $f'(t)$ for $f(\pm\infty) = 0$ to $\mathrm{i}\omega F(\omega)$, $f''(t)$ for $f'(\pm\infty) = f(\pm\infty) = 0$ to $-\omega^2 F(\omega)$.

The complex Fourier transform of (Eq. 10) after rearranging has the following form:

$$\bar{r}_1 \cdot [-m_1 \cdot \omega^2 + \mathrm{i} \cdot b_1 \cdot \omega + k_1] + \bar{r}_2 \cdot [-\mathrm{i} \cdot b_1 \cdot \omega - k_1] = 0$$
$$\bar{r}_1 \cdot [-\mathrm{i} \cdot b_1 \cdot \omega - k_1] + \bar{r}_2 \cdot [-m_2 \cdot \omega^2 + \mathrm{i} \cdot b_1 \cdot \omega + \mathrm{i} \cdot b_2 \cdot \omega + k_1 + k_2] + [-\mathrm{i} \cdot b_2 \cdot \omega - k_2] = 0,$$
$$\bar{F}_{dyn} = \bar{r}_2 \cdot [\mathrm{i} \cdot b_2 \cdot \omega + k_2] + [-\mathrm{i} \cdot b_2 \cdot \omega - k_2]. \tag{11}$$

The first two equations of (Eq. 11) can be written as $[a] \cdot \{\bar{r}\} = \{PS\}$ or in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{Bmatrix} \bar{r}_1 \\ \bar{r}_2 \end{Bmatrix} = \begin{Bmatrix} PS_1 \\ PS_2 \end{Bmatrix} \tag{12}$$

A solution is the found using the Cramer's rule, i.e., $\bar{r}_2 = \dfrac{D_2}{D}$, $\bar{r}_1 = \dfrac{D_1}{D}$, (13)

where $D = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$, $D_1 = PS_1 \cdot a_{22} - PS_2 \cdot a_{12}$, $D_2 = a_{11} \cdot PS_2 - PS_1 \cdot a_{21}$.

If we then consider that in the Fourier transform the parameter ω represents the angular frequency in $\left[\frac{rad}{s}\right]$, then the coefficients a_{ij} in (Eq. 12) have the following form:
$a_{11} = \left(k_1 - m_1 \cdot \omega^2\right) + \mathrm{i} \cdot \left(b_1 \cdot \omega\right)$, $a_{12} = \left(k_1\right) + \mathrm{i} \cdot \left(-b_1 \cdot \omega\right)$, $PS_1 = 0 + \mathrm{i} \cdot 0$, $a_{21} = \left(k_1\right) + \mathrm{i} \cdot \left(-b_1 \cdot \omega\right)$,
$a_{22} = \left(k_1 + k_2 - m_2 \cdot \omega^2\right) + \mathrm{i} \cdot \left(\left(b_1 + b_2\right) \cdot \omega\right)$, $PS_2 = (k_2) + \mathrm{i} \cdot \left(b_2 \cdot \omega\right)$.

The expression (Eq.13) is calculated numerically for chosen values of ω in the selected frequency band. In this solution we ignore the damping of the tire, i.e. $b_2 = 0$ [kg s^{-1}]. The solution thus applies to the simplified model shown in Fig. 3. Since $b_2 = 0$, we have $\overline{F}_{dyn} = k_2 \cdot \left(\bar{r}_2 - 1\right)$.

5 Conclusion

Why do we use transformations? For various reasons, for instance:

- Transformations allow transforming a complicated problem to a potentially simpler one.
- The problem can be then solved in the transform domain.
- Using the inverse transform we obtain solutions in the original domain.
- Fourier transform is appropriate for periodical signals.
  - It allows uniquely transforming a signal from/to time representation $f(t)$ to/from frequency representation $F(\mathrm{i}\omega)$.
  - It allows analyzing the frequency content (spectrum) of a signal (for instance in non-invasive methods – material diagnostics or magnetic resonance).

The basis of every experimental science is measurement, since it is the only tool to quantitatively describe properties of real-world physical processes. Solution of dynamical problems can be realized both in the time and the frequency domain. Both forms have their advantages, complement one another and represent two different facets of the same physical phenomenon.

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NEW TRENDS IN TEACHING STATISTICS

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ABSTRACT. *The paper deals with possibilities of increasing of effectiveness of teaching statistics through modern methods and forms of teaching, such as project based teaching, problem based learning, heuristic methods, use of humor in teaching, etc. We rely on papers published especially in the major journals of the American Mathematical Society - Journal of Statistics Education and The American Statistician.*

KEY WORDS: *statistics education, project based teaching, problem based learning, humor, Journal of Statistics Education.*

CLASSIFICATION: *D40*

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Introduction

Currently, we encounter a decrease in pupils' and students' interest in mathematics education, and the associated reduced level of mathematical knowledge. Therefore, teachers have to strategically plan the course of teaching in a way to motivate students to active acquisition of knowledge. One of the less favorite mathematical disciplines is statistics, which is considered by students to be too hard and considerably boring. The paper deals with modern methods and forms of teaching, which should improve students' attitudes towards statistics and motivate them to gain more fixed knowledge.

Methods and Material

The problems outlined in the introduction encountered by statistics teachers worldwide, what is supported by the nationality of authors of the available literature dealing with the issue. Since 1993, the American Statistical Association has been publishing an electronic Journal of Statistics Education, in which the authors publish their experience with teaching statistics. The archive is available on-line at the web page of the American Statistical Association, the whole lin is: http://www.amstat.org/publications/jse/. The journal deals with forms, methods and evaluation of statistics education. Another source was the journal The American Statistician. The paper lists new trends in teaching mathematical statistics and experience of teachers from different countries. The most used modern methods include project based teaching, problem based learning, heuristic methods and humorous elements in teaching.

Results and Discussion

One of the most common methods implemented in teaching statistics is project based teaching. In a learning project, students have some influence on the choice, respectively a closer definition of the topic. The process of learning through this aspect is characterized by openness. The program of learning is not fixed in all its details before realization of the project. The project concerns extra-curricular activities; studied problems are not based on a textbook, but are present in a region they live in. The project is based on a premise that pupils or students are involved, they work on it on their own accord and that they enjoy the work without extrinsic motivation. Learning outcomes lead to actual results, based on which pupils or students may gain the corresponding knowledge and skills, but also the reward resulting from solution.

Clint W. Coakley from Virginia [2] suggests several helpful approaches in teaching statistics, ideas for classroom presentations, projects, and student-generated data. He believes that one of the goals of statistics courses is for students to understand the importance and usefulness of statistics and to discuss the suitability or unsuitability of the used statistic methods. According to Coakley, the most effective way to help student realize the value of statistic methods is through examples – prospects or pamphlets, web pages often include data which can students analyze using statistical methods, e.g. comparison of home prices for two residential areas. As a part of a project, students may carry out a comprehensive statistical analysis of selected problem including introduction, discussion about the used methods, criticism of presumption, data and conclusions analysis. Projects can be carried out individually or in groups. The author also lists some projects done by his students during statistics education: one-sample tests – lengths of needles from white pine trees, weight of 20 random boys, playing times of 32 CDs with classical music, ages of skydivers from one drop zone; paired sample tests – shrinkage of material after washing and drying, men's and women's haircut prices, computer lab usage in July and August; two-sample tests – burning times of an expensive and inexpensive brand of birthday candles, heights of stairs at northwest and southeast corners of a building; variance analysis – toasting time for four slots of a toaster, growth of three varieties of oak trees, prices of three computer printers from 12 suppliers.

According to Smith [8], students often remain only passive participants in statistics classroom and do not gain experience with problems that may arise in real data collection. Hogg [4] states that better than using in the classroom old but real data is to leave the choice up to students – whether it is searching for available data or creating data of their own. Creating projects provides students with experience in asking questions, problem definition, formulation of hypotheses, designing of experiments and surveys; students collect data, work with errors in measurements, analyze data, and communicate their findings. Snee [9] writes that collection and analysis of data is "the heart" of statistical thinking. Data collection supports learning based on experience and connects the learning process with reality. Bradstreet [1] recommends teaching statistics on a basis of lab seminars. The output of projects in classroom led by G. Smith [8] was a final report ranging from 3 to 5 pages that included explanation of project's objectives, the procedure for obtaining data, charts, graphs and formulation of conclusions. He also places an emphasis on writing style of the report – students could use conversational tune, but had to avoid slang, typographical and grammatical errors, redundancy. After submission of reports students presented their results also orally, using visual aids. This way, students also develop their rhetorical skills. Smith [8] also compared study results of groups he taught in the past in a traditional way and groups he taught through project based teaching, and observed a significant improvement in the success rate during exams.

Sandra Fillebrow [3] also considers the method of project based teaching as a very suitable method in teaching statistics for non-mathematical study programs – sociology and psychology. During the project based teaching, students choose their own topic and collected data, conducted a survey or an experiment, or they used the available data. It was only a basic course in statistics which did not cover testing of hypotheses, but students were guided to look at a problem from the view of factors which can affect values of variables. Data were subsequently analyzed using methods of descriptive statistics, tables and graphs. Sandra Fillebrow states that her students often asked for feedback. Students dealt with, for example, favorite color of jelly beans for men and women, presidential candidate preferences, satisfaction with weight room facilities, lifestyle of students. Some students conducted experiments – for example, they sent mail with and without postal code to six different cities and monitored the time of delivery; they colored socks with different stains and observed the effect of a cheap and branded detergent. Some students used data available on the internet – monthly average temperature and homicide rate, motor vehicle accidents.

College is looking for possibilities of increasing of the effectiveness of teaching also through problem based learning and inquiry based learning. Leigh Lawton [5] recommends teaching testing of statistical hypotheses through problem based learning, for example, using the well-known competition Wheel of Destiny. He describes the situation when Stella Stat launched gambling games. There were 5 numbers on the Wheel of Destiny and a player could place 1 dollar on one of the numbers. In case of a win he earned $ 4.75. We have to decide whether the Wheel of Destiny is fair, based on 50 attempts and frequency of falling of numbers from 1 to 5. Students work on the given problem in groups of 3-5, while if they are using computer they have available the application Wheel of Destiny and if not, they could make their own wheels of destiny and try. She used the activity in the initial course of statistics for college students after completing confidence intervals. It is recommended to leave one class for this exercise.

Michael A. Martin [6] from the Australian National University puts emphasis on the use of analogy and heuristics with a link to the daily life and the development of statistical thinking. He explains the concepts in testing statistical hypotheses through analogy with concepts from the field of law, e.g. null hypothesis = defendant is innocent, alternative hypothesis = defendant is guilty, gathering of data = gathering of evidence, calculation of the test statistics = summary of evidence, assumption that the null hypothesis is true = presumption of innocence, type I error = conviction of an innocent person, type II error = acquittal of a guilty person, high power = high probability of convicting a guilty person.

Psychological approach to teaching statistics refers to the importance of humor in statistics - David L. Neumann, Michelle Hood, and Michelle M. Neumann [7] give an example: a teacher asks the students how many statisticians does it take to change a light bulb and the answer should be one, plus or minus three. Students should be able to find the statistical background of this joke and realize that it is about a confidence interval (of a number of statisticians required to change a bulb). Martin [6] gives examples of motivational images, for example, how far can a tethered dog reach as an analogy to box plot. The digital library of the Consortium for Advancement of Undergraduate Statistics Education (CAUSE) contains in the fun section 130 fairy tales (3 of them animated), 167 quotes, 24 jokes, 20 poems, 69 songs, 7 μ-Tube videos and gallery of 23 statistical arts (https://www.causeweb.org/resources/fun/). Humor in teaching statistics attracts attention, increases interest, but also encourages logical thinking and memory.

Conclusion

This paper summarizes new trends in teaching statistics, which have proved beneficial to statistics teachers especially at universities in different countries. Teaching methods like project based teaching, problem based learning, inquiry based learning and teaching with elements of humor help students to acquire a positive attitude towards statistics, leading to an increased interest in statistics and motivation to gain new knowledge. This is also confirmed by contributions of statistics teachers published in scientific journals. All the above mentioned methods are difficult to prepare and require teacher's creative approach and good organizational skills.

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SOLVING OF GEOMETRICAL PROBLEMS USING DIFFERENT METHODS

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ABSTRACT. *Geometry is used daily by almost everyone. Geometry is found everywhere: in art, architecture, engineering, robotics, land surveys, astronomy, sculptures, space, nature, sports, machines, cars and much more. Geometry has an important role in the mathematics education process, too. When teaching geometry, spatial reasoning and problem solving skills will be developed. In the early years of geometry the focus tends to be on shapes and solids, then moves to properties and relationships of shapes and solids and as abstract thinking progresses, geometry becomes much more about analysis and reasoning. In this article we present six geometrical problems solved using different methods.*

KEY WORDS: *Geometry. Problem solving. Geometrical constructions.*

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Introduction

Simply written, geometry is the study of the size, shape and position of 2 dimensional shapes and 3 dimensional figures. However, geometry is used daily by almost everyone. Geometry is found everywhere: in art, architecture, engineering, robotics, land surveys, astronomy, sculptures, space, nature, sports, machines, cars and much more.

Geometry takes an important place in the mathematics education process too. When teaching geometry, spatial reasoning and problem solving skills will be developed. In the early years of geometry the focus tends to be on shapes and solids, then moves to properties and relationships of shapes and solids and as abstract thinking progresses, geometry becomes much more about analysis and reasoning [1].

Notes to geometrical constructions

At all school levels in Slovakia the experts in mathematics education recognise several types of mathematical problems which can be specified in one term of the characterization.

These types of problems are *word tasks* – tasks in which relationships among given and studied information are expressed in word formulation; *geometrical constructions* – tasks related to geometrical figures and their constructions in accordance with conditions required to outputs.

To the purpose of this article we put attention to geometrical constructions.

To solve these types of tasks is important to use logical rules which are based on the theorems in such way that we achieve the solution from the given elements. The next specification of solution of the geometrical constructions is in extra steps of process of solutions. The steps are analyse, drawing, proof of construction and discussion. Some of these steps we emphasize in problems below.

It is important to note that there are a few methods how to solve the geometrical constructions:

1. *Method of geometrical locus of points in the plane;*

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2. *Method of geometrical transformations in the plane;*
3. *Algebraic–geometrical method;*
4. *Method of vector geometry and coordinates.*

In school practice when teaching geometry we often choose geometrical problems whose solutions need only one method. However, there are many problems which require different methods in different parts of solution. Then we should talk about the solution of the problems using combined methods. For example we use in some part of the solution the method of geometrical transformation and in other part we solve the tasks using algebraic-geometrical method and after it we use constructional sequences. In this article we show solutions of six geometrical problems in whose solutions we use different of the methods mentioned above.

Selected geometrical problems and their solutions using different methods

Problem 1

On the sides BC, CA and AB of triangle ABC are lying points A', B' and C'. Let points A_1, B_1, C_1 and A_2, B_2, C_2 are images of points A, B, C under the enlargements with the same scale factor k and the centres of enlargements are points C', A', B' and B', C', A'.

Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ have common centroid.

Solution

We draw the triangles pertain to this task (Figure 1). As the scale factor k we choose the number $k = -1$. This choice does not upset the generality of the proof.

For prove we choose the vector method.

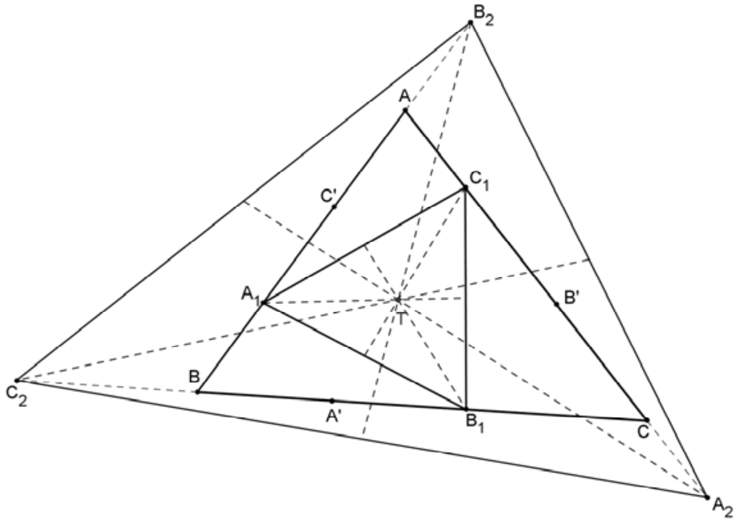

Figure 1

For the points $A', B', C', A_1, B_1, C_1$ and A_2, B_2, C_2 we can write

$$\overrightarrow{C'A_1} = k \cdot \overrightarrow{C'A}, \quad \overrightarrow{A'B_1} = k \cdot \overrightarrow{A'B}, \quad \overrightarrow{B'C_1} = k \cdot \overrightarrow{B'C}, \qquad \text{(a)}$$
$$\overrightarrow{B'A_2} = k \cdot \overrightarrow{B'A}, \quad \overrightarrow{C'B_2} = k \cdot \overrightarrow{C'B}, \quad \overrightarrow{A'C_2} = k \cdot \overrightarrow{A'C}.$$

Label the centroids of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ as points T_1 and T_2.

Then holds true that:

$$\overrightarrow{T_1A_1} + \overrightarrow{T_1B_1} + \overrightarrow{T_1C_1} = \vec{0}, \qquad (1)$$
$$\overrightarrow{T_2A_2} + \overrightarrow{T_2B_2} + \overrightarrow{T_2C_2} = \vec{0}. \qquad (2)$$

From these implies that

$$\overrightarrow{T_1A_1} = \overrightarrow{T_1C'} + \overrightarrow{C'A_1} = \overrightarrow{T_1C'} + k \cdot \overrightarrow{C'A}$$
$$\overrightarrow{T_1B_1} = \overrightarrow{T_1A'} + \overrightarrow{A'B_1} = \overrightarrow{T_1A'} + k \cdot \overrightarrow{A'B}$$
$$\overrightarrow{T_1C_1} = \overrightarrow{T_1B'} + \overrightarrow{B'C_1} = \overrightarrow{T_1B'} + k \cdot \overrightarrow{B'C}$$
$$\overrightarrow{T_2A_2} = \overrightarrow{T_2B'} + \overrightarrow{B'A_2} = \overrightarrow{T_2B'} + k \cdot \overrightarrow{B'A}$$
$$\overrightarrow{T_2B_2} = \overrightarrow{T_2C'} + \overrightarrow{C'B_2} = \overrightarrow{T_2C'} + k \cdot \overrightarrow{C'B}$$
$$\overrightarrow{T_2C_2} = \overrightarrow{T_2A'} + \overrightarrow{A'C_2} = \overrightarrow{T_2A'} + k \cdot \overrightarrow{A'C}.$$

Using (1) and (2) we obtain

$$\overrightarrow{T_1C'} + \overrightarrow{T_1A'} + \overrightarrow{T_1B'} = k \cdot \left(\overrightarrow{C'A} + \overrightarrow{A'B} + \overrightarrow{B'C}\right) \qquad (1')$$
$$\overrightarrow{T_2B'} + \overrightarrow{T_2C'} + \overrightarrow{T_2A'} = k \cdot \left(\overrightarrow{B'A} + \overrightarrow{C'B} + \overrightarrow{A'C}\right) \qquad (2').$$

For the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ holds $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ and we derive

$$\overrightarrow{AC'} + \overrightarrow{C'B} + \overrightarrow{BA'} + \overrightarrow{A'C} + \overrightarrow{CB'} + \overrightarrow{B'A} = \vec{0}.$$

This means that $\overrightarrow{C'A} + \overrightarrow{A'B} + \overrightarrow{B'C} = \overrightarrow{A'C} + \overrightarrow{B'A} + \overrightarrow{C'B}$.

Now from (1´) and (2´) follows

$$\overrightarrow{T_1C'} + \overrightarrow{T_1A'} + \overrightarrow{T_1B'} = \overrightarrow{T_2B'} + \overrightarrow{T_2C'} + \overrightarrow{T_2A'}$$

That means the points T_1 and T_2 cincide in one point T (see Figure 1).

Problem 2

Divide a segment *AB* into two segments where the ratio of the length of given segment to the length of the longer part is the same as the ratio of the length of the longer part to the length of the shorter part.

Solution

Concider a point C

Let mark the dividing point as $C \in AB$ (as a point on segment *AB*) such that $|AC| > |BC|$. Let $|AB| = a$, $|AC| = x$, then $|BC| = a - x$ (Figure 2).

The problem in algebraical expression is in a form

$$a : x = x : (a - x)$$

Solving

$$x^2 = a \cdot (a - x)$$
$$x^2 + ax - a^2 = 0 \qquad (1)$$

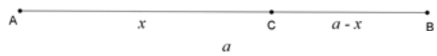

Figure 2

The roots of the equation (1) are

$$x_1 = \frac{1}{2}a\left(\sqrt{5} - 1\right) \qquad (2)$$
$$x_2 = \frac{1}{2}a\left(-\sqrt{5} - 1\right) \qquad (3)$$

The second root is negative that does not satisfy the task.

To construction of the point *C* we modify the expression (2)

$$x_1 = \frac{1}{2}a(\sqrt{5}-1) = \sqrt{a^2 + \left(\frac{1}{2}a\right)^2} - \frac{1}{2}a$$

The segment of length $\sqrt{a^2 + \left(\frac{1}{2}a\right)^2}$ can be constructed using by Pythagorean´ theorem (Figure 3).

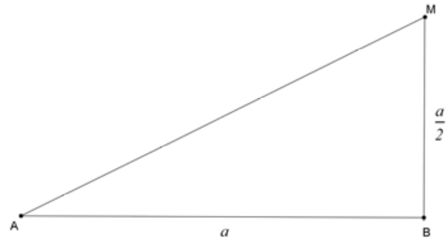

Figure 3

Then we draw segment *x* as we see in Figure 4.

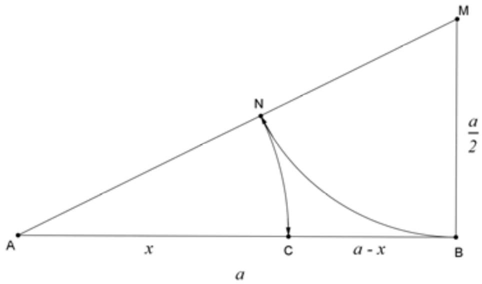

Figure 4

Therefore is valid

$$|BM| = \frac{1}{2}a,$$
$$|MN| = |BM| = \frac{1}{2}a,$$
$$|AM| = \sqrt{a^2 + \left(\frac{1}{2}a\right)^2} = \frac{a}{2}\sqrt{5},$$
$$|AC| = |AN| = |AM| - |MN| = \frac{a}{2}\sqrt{5} - \frac{1}{2}a = \frac{a}{2}(\sqrt{5}-1),$$
$$|BC| = |AB| - |AC| = a - \frac{a}{2}(\sqrt{5}-1) = \frac{a}{2}(3-\sqrt{5}).$$

Now we should express the relevant ratios:

$$\frac{|AB|}{|AC|} = \frac{a}{\frac{a}{2}(\sqrt{5}-1)} = \frac{2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{1+\sqrt{5}}{2} = \varphi,$$
$$\frac{|AC|}{|BC|} = \frac{\frac{a}{2}(\sqrt{5}-1)}{\frac{a}{2}(3-\sqrt{5})} = \frac{\sqrt{5}-1}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{1+\sqrt{5}}{2} = \varphi.$$

We used a combined process during the solution of this task. This combined process we call algebraic – geometric method. This construction originates from Heron.

Note: The ratio φ is called as golden number. We say that the line segment AB is dividing according to the golden ratio.

Problem 3

Let be given the line *m*, the length *a*, angle α and a point *V*. Construct a triangle *ABC* in conditions that its side $BC \subset m$, $BC = a$ and $\sphericalangle A = \alpha$.

Solution
Analyse: Let us scatch the triangle *ABC* (we assume that such triangle exists).

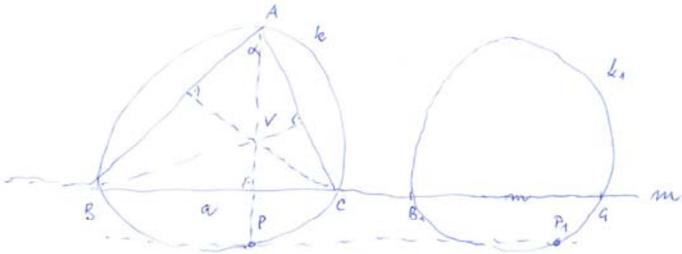

Figure 5

Let k be circumcircle of a the triangle *ABC*. On the circle k there are the points reflected with point V under the reflection of sides of triangle *ABC*. One of these points we label as *P*. This point *P* we could construct from the given objects. The points *A, V, P* are collinear and their line is perpendicular to line *m*. The point *A* is the vertex of the angle (of given value α) at the circumference to the circle *k* pertain to the chord *BC*. The most important is the construction of the circle *k* which helps us to construct all of the vertices of the triangle *ABC*.
Let a translation of the triangle *ABC* which has been translated the side *BC* to B_1C_1 lying on line m. Then the point *P* is translated to P_1 lying on line *m´*. Line *m´* is parallel to line *m* with the circle k has been translated to the circle k_1. The circle k_1 contain the locus of the points at which a line segment *BC* subtends an angle α. Let us concentrate on the construction of the circle k_1.

Construction
1. B_1C_1; $|B_1C_1| = a$
2. k_1; k_1 is the set of all points $X \in p$ such that $|\angle B_1XC_1| = \alpha$
3. P_1; $P_1 \in k_1 \cap m', m' \parallel m, P \in m'$
4. k; $T_{\overrightarrow{P_1P}}: k_1 \to k$
5. $B, C; B, C \in k \cap m$
6. $A; A \in k \cap \overleftrightarrow{VP}$
7. $\Delta\, ABC$

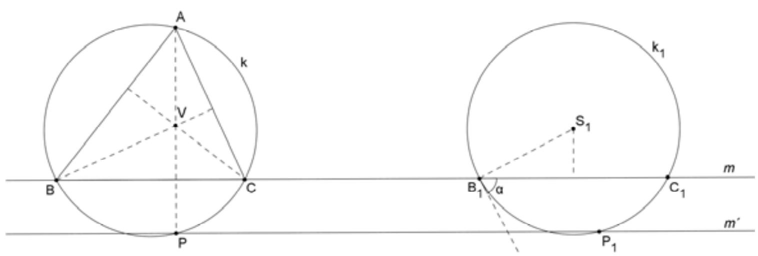

Figure 6

The number of solutions depends of the intersection of the line *m´* and the circle k_1.

Problem 4

Let a triangle ABC. Let points M_1 and M_2 situated on the line BC, $N^{\prime}$ is any point situated on the side CA, $P^{\prime}$ is any point situated on the side AB. Mark the intersection points of the lines $N^{\prime}M_1$ and $N^{\prime}M_2$ with side AB as Q_1 and Q_2, the intersection points of the lines $P^{\prime}M_1$ and $P^{\prime}M_2$ with side AC as R_1 and R_2. Prove that the lines BC, Q_1R_2, Q_2R_1 cross each other in one point.

Solution

It should be to prove that three lines are concurrent in one point, we prefer to use Menelaus' theorem. The points R_2,Q_1 and I are lying on one line and on the lines containing the sides of the triangle *ABC,* according to Menelaus´ theorem holds

$$(BCI) \cdot (CAR_2) \cdot (ABQ_1) = 1 \qquad (1)$$

Analogically it is true

$$(BCJ) \cdot (CAR_1) \cdot (ABQ_2) = 1 \qquad (2)$$

From the collinearity of points we obtain similarly:

$$(BCM_1) \cdot (CAN^{\prime}) \cdot (ABQ_1) = 1 \qquad (3)$$
$$(BCM_2) \cdot (CAN^{\prime}) \cdot (ABQ_2) = 1 \qquad (4)$$
$$(BCM_1) \cdot (CAR_1) \cdot (ABP^{\prime}) = 1 \qquad (5)$$
$$(BCM_2) \cdot (CAR_2) \cdot (ABP^{\prime}) = 1 \qquad (6)$$

From (3), (6) and (4), (5) we derive

$$(CAN^{\prime}) \cdot (ABP^{\prime}) \cdot (BCM_1) \cdot (BCM_2) \cdot (CAR_2) \cdot (ABQ_1) = 1 \qquad (7)$$
$$(CAN^{\prime}) \cdot (ABP^{\prime}) \cdot (BCM_1) \cdot (BCM_2) \cdot (CAR_1) \cdot (ABQ_2) = 1 \qquad (8)$$

From (7) and (8) it follows that

$$(CAR_2) \cdot (ABQ_1) = (CAR_1) \cdot (ABQ_2);$$

From there and (1), (2) it implies that $(BCI) = (BCJ)$. It means that the points *I* and *J* merge into one point and therefore the lines BC, Q_1R_2, Q_2R_1 cross each other in one point.

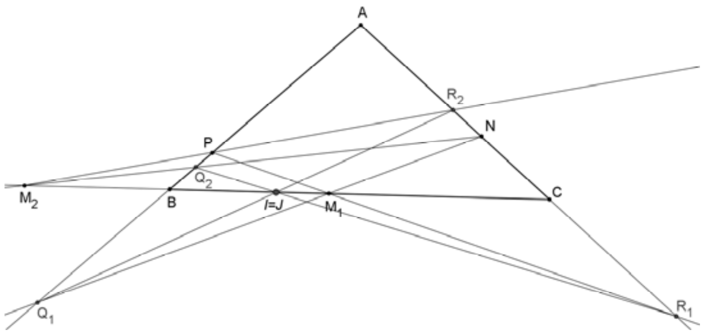

Figure 7

Problem 5

Let line segments of length *a, b, e, f* and an angle of value ε. Construct a quadrilateral *ABCD* where $|AB| = a$, $|CD| = c$, $|AC| = e$, $|BD| = f$ and $|\angle ASB| = \varepsilon$, while *S* is the intersection point of the quadrilateral´s diagonals.

Solution

Analyse: We assume that there exists a quadrilateral *ABCD* suitable to the task. Mark the intersection of the diagonals as *S*. In translation $D \rightarrow C$ the point *B* move to point *B´*, the diagonal *BD* move to segment *CB´* parallel to *DB*, then $|\angle ASB^{\prime}| = \varepsilon$.

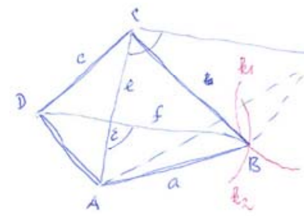

Figure 8

Thus we have concluded: If the quadrilateral has the required properties, then for triangle $AB'C$ we have $|AC| = e$, $|CB'| = f$ and $|\angle ASB'| = \varepsilon$. Based on these information the triangle AB´C is constructible. Then the point B lies on the circles $k_1(B'; c)$ and $k_2(A; a)$. Point D we get in translation $B' \to B$ from point C.

Construction

1. $\Delta\, AB'C$, $|\angle ACB'| = \varepsilon$, $|AC| = e$, $|CB'| = f$
2. k_1; $k_1(B'; c)$
3. k_2; $k_2(A; a)$
4. B; $B \in k_1 \cap k_2$
5. D; $T_{\overrightarrow{B'B}}: C \to D$
6. $ABCD$

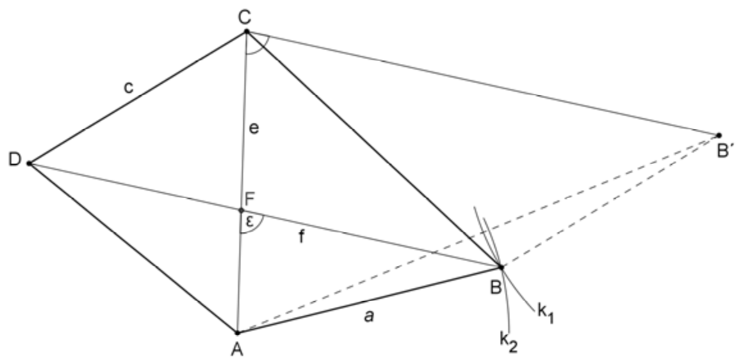

Figure 9

The number of solutions depends of the intersection of the circles k_1, k_2.

Problem 6

Let triangle *ABC*. Construct on its sides points X, Y where $|AX| = |XY| = |YC|$.

Solution

There is not written what kind of triangle is *ABC*. First we solve the task for isosceles triangle *ABC*.

From the symmetry of triangle *ABC* under the perpendicular bisector of the base *AC* follows the parallelism $AC \parallel XY$. The line *AY* intersects the lines *AC*, *XY* under the same alternate angles α. The triangle *AYX* is an isosceles triangle with a base *AY*, the angles connected to the base are congruent. The line *AY* is the bisector of the angle *BAC*.

Construction (Figure 10)
1. $\Delta\, ABC$
2. *m; m* is the bisector of angle *BAC*
3. Y; $Y \in m \cap BC$
4. X; $X \in AB$ and $XY \parallel AC$

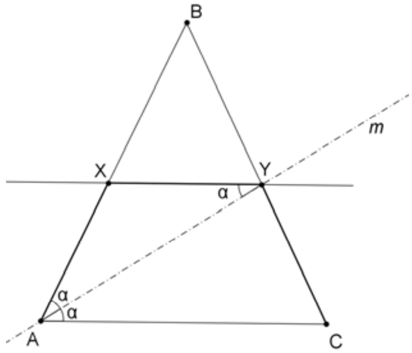

Figure 10

Now we think about a general triangle *ABC*.
We assume that there exist the points *X, Y* suitable to the task.

The points *X, Y* we cannot construct directly, we can construct the points *X′, Y′ C′* in enlargement with the centre *A*. The point *X′* is … lying on *AB* because the scale factor is not done. For the point *Y′* is true: $Y' \in k(X', |AX'|)$ and $Y' \in p$, where p is a parallel to line AC and point $\bar{Y}$ lies on the line p where $\bar{Y} \in BC$ and $|\bar{Y}C| = |AX'|$. The points *X* and *Y* we will found as images of points *X′* and *Y′* under the enlargement with the centre *A* which move the point *C′* to point *C*.

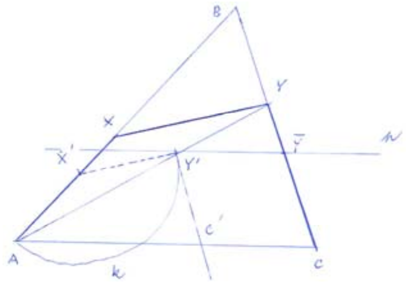

Figure 11

Conclusion

Geometry in general does not belong in students′ opinion to favourite content of mathematics [7]. For students is this area of curriculum very difficult, especially geometrical constructions. That reason calls for more methods to use in solving geometrical problems. Another reason could be as Lester [8] claims „Good problem solvers tend to be more concerned than poor problem solvers about obtaining „elegant“ solutions to problems.

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QUALITATIVE RESEARCH OF PROBABILITY TEACHING WITH DIDACTIC GAMES

EVA UHRINOVÁ

ABSTRACT. *The following paper gives several conclusions from a qualitative research focused on observation of performance and behavior of pupils during lessons in which were used didactic games. The observation was conducted on fourteen year old pupils (13 girls, 6 boys) from the Sports Gymnasium of J. Herda in Trnava, from September 2013 to March 2014.The observation was conducted on 28 lessons where were tested 30 didactic games focused on teaching of the thematic area Combinatorics, probability, statistics. We used the method of direct observation, i.e. the studied reality was observed in the field. The paper especially focuses on what pupils understand under the concept of the game, how they perceive teaching through play, what types of games pupils enjoy and what kind of knowledge of the thematic area Combinatorics, probability, statistics did pupils acquire through play. The paper also mentions opinions of pupils on teaching with games, which they expressed in a questionnaire filled in at the end of the qualitative research.*

KEY WORDS: *observation, didactic games, combinatorics, probability, statistics.*

CLASSIFICATION: *A20, D40, K10*

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Introduction

The latest school reform puts more emphasis on teaching of the thematic area of combinatorics, probability and statistics. This thematic area does not belong to teachers' favorites and thus it is also not very popular for pupils. Lately, the recommended number of lessons in elementary schools dealing with these topics has been doubled when compared with the curriculum from the year 1997, and the knowledge should be acquired spirally in each grade throughout the whole school year. The state education program recommends addressing a number of topics using games [2]. We have created didactic games from the thematic area Combinatorics, probability, statistics and verified their use in the classroom.

The following paper presents conclusions of a qualitative research aimed at monitoring of pupils' performance and behavior during the lessons in which were used didactic games.

Implementation of the Qualitative Research

In accordance with Gavora, we understand monitoring as observation of human activity, record (registration or description) of this activity, its analysis and evaluation[1].

The monitoring was carried out in the grade 8 classroom (13 girls, 6 boys) of the Sports Gymnasium of J. Herda in Trnava in Slovakia, from September 2013 to March 2014. We observed 28 lessons and tested 30 didactic games. We used the method of direct observation, i.e. the studied reality was observed in the field. According to the method of recording we carried out a structured observation.

This type of observation is best used for activities that are in some way organized, not spontaneous, such as activity of pupils during lesson. When using the structured observation we are looking for answers to pre-defined and designed phenomena[3].

We have set the following goals of the qualitative research and research questions:

- To find out how the pupils perceive teaching through games.
  - What the pupils understand under the concept game?
  - Do pupils with lower grades like the game?
  - What types of games do pupils enjoy/don't enjoy?
  - What requirements does a game have to meet to be enjoyable for pupils?
- To find out the knowledge of pupils of the thematic area Combinatorics, probability, statistics.
  - Do the pupils have the necessary knowledge of the thematic area Combinatorics, probability, statistics according to the State Education Program?
  - Is it possible to teach the thematic area Combinatorics, probability, statistics through games?
  - Did the pupils acquire some knowledge of the thematic area Combinatorics, probability, statistics when using games?

There was only one teacher (the author of the article) present on the lesson. The observed reality was recorded in the observation sheet through short notes and more detailed notes after the lesson. During lessons we recorded the occurrence of phenomena (frequency) by natural coding of the observed phenomena in the observation sheet with specified basic observed categories of phenomena, thus the code was noted if the phenomenon occurred, and at the same time were noted some pupils' comments and short notes. After the end of the lesson, detailed notes about what was happening in the classroom were written down. To facilitate the observation entries in some categories we used observation scales. Pupils expressed their opinion on teaching with games also through a questionnaire which was given to them at the end of March 2014.

Some Conclusions from the Observations

How pupils understand the concept of game

Pupils' answers on the item in the questionnaire: Explain what do you understand under the concept of game:

- "Game is a work that we like to do"
- "A group of people playing together"
- "Game is fun." – this response occurred several times.

We were surprised by one pupil's answer and we present it in the fig. 1: "Activity where we can unleash our potential and show what we know. We have the opportunity to express ourselves and to have fun at the same time."

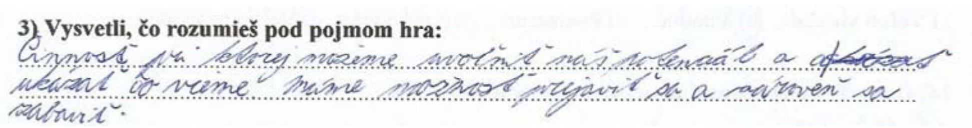

Figure 1: Pupils on games

Based on the responses of pupils in the questionnaire, but mainly on experience from observing pupils during lessons where were used didactic games, but also during common lessons, we come to the conclusion that pupils consider a game to be **any activity** during lesson where they work in an unusual way, e.g. filling in crosswords or solving of problems given in an unusual way. For example, if pairs compete against each other in solving crossword.

How pupils perceive teaching with games

Pupils see the inclusion of games into the learning process as fun and relax, not learning. They feel that they have a free period, they can move freely around the classroom, they can talk, have fun.

Pupils had initially felt that a game is a voluntary activity and if they don't want to, they don't have to get involved in the game, respectively, they can end and discontinue the game at any time.

After the game, the teacher led a discussion about the game, phenomena, knowledge which pupils could learn through the game. In several cases the teacher tried to use the game as a motivation to explain some information. However, it was difficult to teach pupils to discuss the game. From the pupils' perspective, the game was over and they did not want to discuss it any further. They did not engage in discussions and they rarely remembered any conclusions. Some pupils were not able to memorize the relationship to calculate the probability even after the third game. At first, pupils did not use findings from one game in another. They saw each game as a new activity and did not see the connection. However, over time, some of them did learn to see it.

Pupils do not see the opportunity to learn something new, to explore or to find something out as a motivation; a game has to contain an interesting activity, aids, or the possibility to compete with others.

63% of pupils consider teaching mathematics through games to be very interesting (Fig. 2).

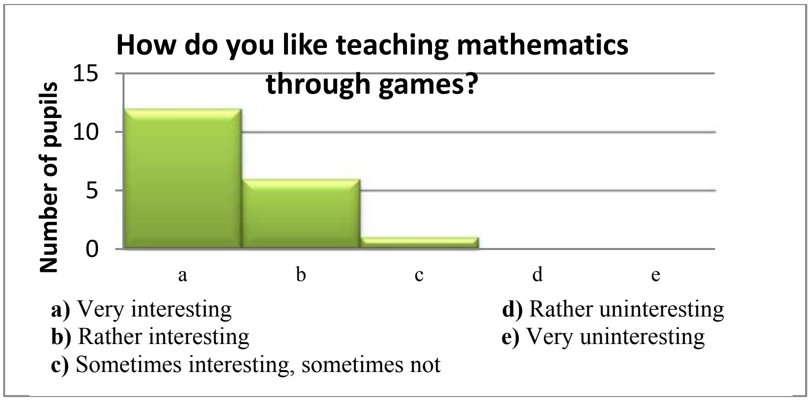

Figure 2: Pupils' responses on an item in the questionnaire

Although the pupils gradually learned to discuss the games and found out that we played games concentrated on the thematic area Combinatorics, probability, statistics and that they are supposed to gain some knowledge from this area through playing, they did not take the inclusion of games as learning but a free period.

In the questionnaire they were supposed to express their attitude towards the statement: "When we play a game we have a free period." Up to 68% of pupils expressed agreement or strong agreement with this statement (Figure 3).

Figure 3: Pupils’ responses on an item in the questionnaire

What types of games pupils enjoy

Pupils especially enjoyed group games. Creating of groups was affected by social relationships between pupils. Some girls refused to be in a group with boys and vice versa. If it was the teacher who divided the class into groups, some pupils got angry and refused to cooperate and participate in the game.

Majority of pupils (52,63%) prefers working in a group of 4 (Figure 4).

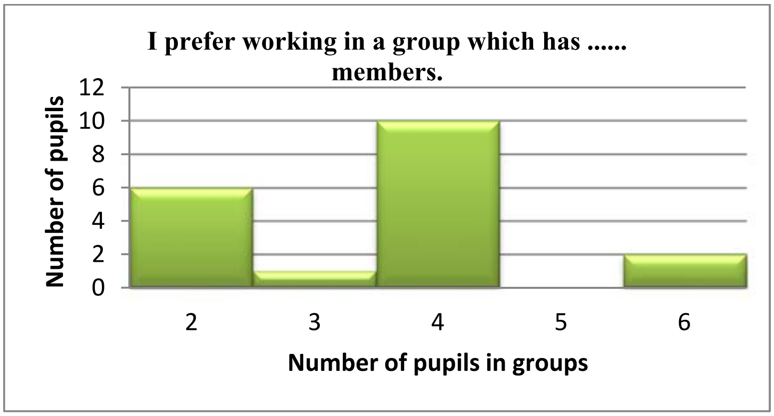

Figure 4: Pupils’ responses on an item in the questionnaire

Pupils justified their choice of four-member groups in the questionnaire as follows:

- If a group is smaller, everyone has a chance to get involved
- As we help each other, we complete one another
- More “fun”
- More brainiacs
- We can choose the best from several opinions
- We have more opinions
- I hardly have to do anything

Six pupils preferred working in pairs and their reasons are:

- It is easier to reach an agreement

- One is enough
- There are few of us, we don't have to shout at each other

Pupils liked the most games of chances, where the outcome depended on chance, i.e. they did not have to think during the game.

Teaching of the thematic area Combinatorics, probability, statistics through games

We worked on this thematic area in spiral, twice a month the teaching of mathematics included games designed for this thematic area. Each game was followed by a discussion.

Games were aimed at **understanding of the concepts of random experiment, random event, odds, probability, certain, impossible, probable event.**

Pupils had problems using these concepts in games. They argued that they are familiar with these concepts, but were not able to use them; they did not understand them well. In several cases they marked a low-probability event as impossible respectively interpreted low probability as zero. The teacher was constantly using these terms during the games. After about the tenth game, pupils gradually began to use them as well, thus, they commented that an event with zero probability was impossible and were able to distinguish a game of chance.

Pupils perceived the concepts of **odds and probability** as semantically identical. They felt that probability is just a professional term for odds. They were able to calculate the probability of a random experiment with only two different results. For example, they were able to say that the odds of the outcome heads when tossing a coin is 1:1 and that the probability is 50%. They were not able to calculate the probability of an event with several outcomes of a random experiment, like when throwing a dice. However, they were gradually able to evaluate and compare chances of winning of two players.

Pupils dislike using decimal numbers and thus did not like probability in the form of decimals. They understood the outcome in the form of percentage better.

Although the pupils gradually learned to calculate the probability in majority of cases they still made their decisions based on luck, hunch. In games they were choosing an option according to their hunch bit according to the likeliness of an event.

We found out that pupils have only poor understanding of **the concept of randomness**. Some pupils (regardless of their grades) think that the desired outcome of a random experiment depends on luck. During the game Draw a person they commented on their failure to draw a white ball from the pouch with four black and one white that they were not lucky that day.

The games were also aimed at understanding of estimation of probability using relative frequency. Pupils were not familiar with the **frequency view of probability;** they didn't encounter calculating a relative frequency of an event. Thanks to the games, pupils themselves realized that they were not able to tell anything about the probability of an event after conducting one random experiment, but that if they conduct enough experiments, they are able to fairly accurately estimate the probability of the event.

The games also focused on **finding a number of all and favorable outcomes of a random experiment.** Pupils had no problems with writing of all options and were also able to construct and use a tree diagram. Proper manipulation with cards during the game Numerical mania enabled the majority of pupils to discover the relationship to calculate the number of variations with and without repetition (of course without using factorials). Pupils who did not manipulate the cards in a proper way but were able to list all the possibilities of the game and to draw a tree diagram did not discover this relationship.

Pupils had problems **analyzing the solutions of games.** They are used to tasks where they take all given data and make several algebraic operations. They made too rapid conclusions when analyzing the games. They made their decisions according to one event and did not check the other. If a pupil found out that there was a ball number six in the pouch, he selected the pouch as a drawing tool, because he was immediately able to come up with several outcomes of a random experiment where he win with the number six. However, there were also balls with other numbers in the pouch, but the pupil did not care. Only pupils with higher grades were able to learn to properly analyze the game and find all possible outcomes of the random experiment.

Conclusion

This paper presents several conclusions from the observation of lessons taught using didactic games. It mentions pupils' understanding of the concept of a game, their perception of teaching and learning through games, which types of games do they prefer and what knowledge of the thematic area Combinatorics, probability, statistics did they gain through the games. Other conclusions with more detailed information on the conducted research will be found in the dissertation thesis of the author. Pupils had little knowledge of the thematic area Combinatorics, probability, statistics.

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NOTES ON SOLUTION OF APOLLONIUS' PROBLEM

DUŠAN VALLO[1], VILIAM ĎURIŠ

ABSTRACT. *In our paper we concerned with a problem of Apollonius which is signed as CCC in symbolic way in relevant sources. The idea of finding a solution is based on using specific circle inversion in manner which is not conventional. An original construction of the center of inversion causes that inversion maps given circles into three circles of equal radii. This approach substantially simplifies a solution. Some little methodological instructions to using of dynamical software are implemented, too.*

KEY WORDS: *Apollonius' problem, circle inversion, pencils of circles, radical axis*

CLASSIFICATION: *G45, G55, G95*

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Introduction

In our article we discuss one specific solution of the last Apollonius' problem how to draw a circle that is tangent to three given circles in a general position (the circles are not intersecting and none of them is inside the other).

We apply a peculiar method of the circle inversion in a combination with a method of geometrical loci of points. We suppose that the reader is familiar with fundamentals of non-linear mapping such circle inversion and its properties. Further we assume that the reader is acquainted with geometrical concept of radical axis, power of the point with respect to circle, pencils of circles, algebraic equation of circle and finally definition of hyperbola.

The Tangency Problem of Apollonius – short overview

The Problem of Apollonius is one of the most famous geometric problems which was put forth by the greatest scientist of ancient world, Apollonius of Perga (ca. 260 – 170 B.C.) We can formulate it as following:

Consider the problem how to draw a circle subject to three conditions taken from among the following: the circle pass through one or more points, P; to be tangent to one or more lines, L; to be tangent to one or more circles, C. [1].

There are ten problems that can be solved and symbolically are represented by following $PPP, PPL, PLL, LLL, PPC, PLC, PCC, LLC, LCC$ and CCC.

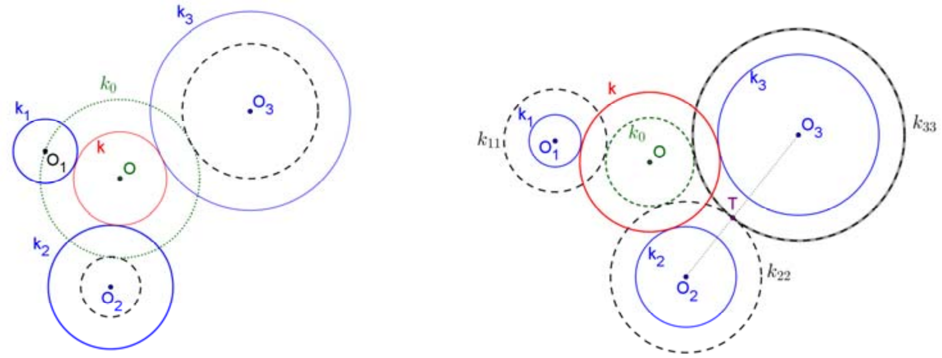
The last one is considered as the hardiest construction problem in general. The *CCC* problem has at most eight possible solutions which drawing can be obtained with different approaches. The methods of finding a solution have been developed by various mathematicians through history, e.g. F. Viete (1540 - 1603), C. F. Gauss (1777 - 1855) or J. D. Gergonne (1771 - 1859). Gergonne's solution is one of the most elegant and is based

[1] Corresponding author

on properties of homothetic circles. The solution is to much extensive for the purpose of this article and so we refer a reader to [2].

Widely used is the method of non-linear mapping called circle inversion. A lot of solutions of the general problem in CCC variant can be reduced to a simpler construction in many ways.

In Fig.1 we suppose that for the given circles $k_1(O_1,r_1)$, $k_2(O_2,r_2)$ and $k_3(O_3,r_3)$ holds $r_1 < r_2 < r_3$. If we draw concentric circles $k_{22}(O_2,r_2-r_1), k_{33}(O_3,r_3-r_1)$ then the circle $k_0(O,r_0)$ is concentric with a circle k which is a solution. This solution is famous like Viete's reduction of the problem to simpler PCC variant. [3] We note that to find all solutions is necessary to consider with further possibilities of the concentric circles such $k_{22}(O_2,r_2 \pm r_1), k_{33}(O_3,r_3 \pm r_1)$ [4].



a) b)

Figure 1: a) Viete's reduction the variant *CCC* to simpler construction problem *PCC*, b) if the center of inversion lies in the point *T* then it leads to a reduction to *PPC* variant.

In [4] one can find a solution in which we draw circles $k_{11}(O_1,r_1+s), k_{22}(O_2,r_2+s), k_{33}(O_3,r_3+s)$, where $s = \frac{|O_2O_3| - (r_2+r_3)}{2}$. In this case a circle $k_0(O,r_0)$ is concentric with a circle k. The simplification resides in a moment of the choice of the center of inversion. If it is a tangency point T of the circles k_{22}, k_{33} then the inversion maps the circles k_{22}, k_{33} onto two parallel lines and the problem is reduced to PPC variant.

In [5] is also described a special construction of circle inversion which maps two non-intersecting circles into a pair of concentric circles. Using this inversion one can map the given three circles $k_1(O_1,r_1)$, $k_2(O_2,r_2)$, $k_3(O_3,r_3)$ into two concentric circles $k_1'(O_1',r_1)$, $k_2'(O_2',r_2')$ and the circle $k_3'(O_3',r_3')$ laying between the circles k_1' and k_2'. To draw a circle k' that is tangent to three circles k_1', k_2', k_3' is a trivial construction task. The circle inversion maps the circle k' into the solution – circle k.

Finally, there are also many other approaches how to solve the problems of Apollonius, including algebraic calculations. There is no place to explain these kinds of solutions in details. We refer the reader to [2, 3, 6, 7, 8, 9].

Theoretical framework

A solution is based on following statement.

Theorem. *The loci of points as centers* S *of circle inversion* $K_i\left[\omega(S,r)\right]$ *which transforms given circles* $k_1(O_1,r_1)$, $k_2(O_2,r_2)$, $O_1 \neq O_2$, $r_1 \neq r_2$ *into circles* $k_1'(O_1',r_0)$, $k_2'(O_2',r_0)$, $O_1' \neq O_2'$ *are two circles* $m_e(O_e,r_e)$ *and* $m_i(O_i,r_i)$. *The points* O_e, O_i *are the centers of homothety of the given circles* $k_1(O_1,r_1)$ *and* $k_2(O_2,r_2)$. [10]

Proof. If an inversion $K_i\left[\omega(S,r)\right]$ maps the circle $k_1(O_1,r_1)$ into a circle $k_1'(O_1',r_0)$ then holds

$$|SO_1'| = \frac{|SP'|+|SQ'|}{2} = \frac{r^2\left(|SP|+|SQ|\right)}{2|SP|.|SQ|} = \frac{r^2}{|SP|.|SQ|}\cdot\frac{|SP|+|SQ|}{2} = \frac{r^2}{|SP|.|SQ|}\cdot|SO_1|,$$

where $P,Q \in O_1O_1' \cap k_1$ and P',Q' their inversion images (Fig. 2)

The product $|SP|\cdot|SQ|$ represents a power h of the point S with respect to the circle $k_1(O_1,r_1)$ [2] and it is easy to derive that

$$\frac{|SO_1'|}{|SO_1|} = \frac{r^2}{|SP|.|SQ|} = \frac{r^2}{(u_1+r_1).(u_1-r_1)}, \text{ where } u_1 = |SO_1|.$$

If the point S lies outside the circle k_1 then the ratio $|SO_1'|:|SO_1|$ is positive and holds

$$\frac{|SO_1'|}{|SO_1|} = \frac{r^2}{|SP|.|SQ|} = \frac{r^2}{(u_1+r_1).(u_1-r_1)} = \frac{r^2}{u_1^2-r_1^2} = \frac{r^2}{h_1}$$

If the point S is an internal point of circle k_1 then $u_1 - r_1 < 0$ and holds

$$\frac{|SO_1'|}{|SO_1|} = \frac{r^2}{|SP|.|SQ|} = \frac{r^2}{(r_1+u_1).(r_1-u_1)} = \frac{r^2}{r_1^2-u_1^2} = -\frac{r^2}{h_1}.$$

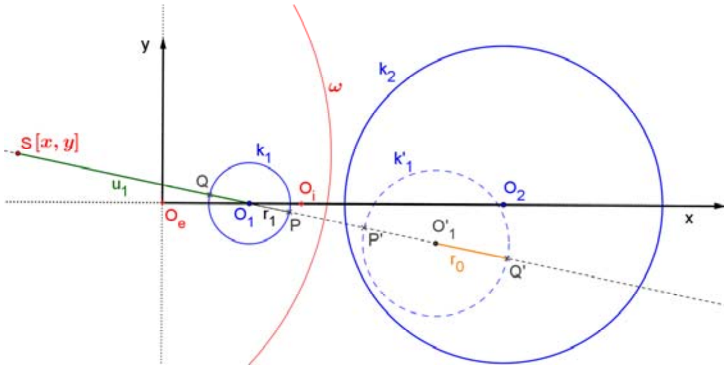


Figure 2 An inversion $K_i\left[\omega(S,r)\right]$ maps the circle $k_1(O_1,r_1)$ into a circle $k_1'(O_1',r_0)$.

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[2] The power of a point P with respect to a circle $k(O,r)$ is $h = u^2 - r^2$, $u = |PO|$. By this definition, points inside the circle have a negative power, points outside the circle have positive power and the points on the circle have zero power. [11]

The circles $k_1(O_1, r_1)$, $k_1'(O_1', r_0)$ are also homothetic with the center S and holds true that $|SO_1'| : |SO_1| = r_0 : r_1$. We obtain a particular result $|SO_1'| : |SO_1| = r_0 : r_1 = r^2 : |h_1|$.
By analogy we derive $r_0 : r_2 = r^2 : |h_2|$, where h_2 is the power of point S with respect to the circle $k_2(O_2, r_2)$. From the previous we have $r_1 : r_2 = \pm h_1 : h_2$.

Let us consider O_e as an external center of homothety of the circles $k_1(O_1, r_1)$, $k_2(O_2, r_2)$. From the properties of the homothety $a_1 : a_2 = r_1 : r_2$, where $|O_e O_1| = a_1$, $|O_e O_2| = a_2$ and we obtain result $a_1 h_2 = \pm a_2 h_1$, $h_j = u_j^2 - r_j^2, j = 1,2$.
We put the origin of Cartesian coordinate system into the point O_e in such manner that $O_1[a_1, 0]$, $O_2[a_2, 0]$, $a_2 > a_1 > 0$.

a) Assume that the point S as an external point of the circles k_1, k_2 or is an internal point of the both of them. Holds true that $a_1 h_2 = a_2 h_1$. If we label $|SO_2| = u_2$ then we derive

$$0 = a_1 h_2 - a_2 h_1 = a_1 \cdot \left(u_2^2 - r_2^2\right) - a_2 \cdot \left(u_1^2 - r_1^2\right) =$$
$$= a_1 \cdot \left[\left(x - a_2\right)^2 + y^2 - r_2^2\right] - a_2 \cdot \left[\left(x - a_1\right)^2 + y^2 - r_1^2\right],$$

what can be simplified $$x^2 + y^2 = a_1 a_2 - r_1 r_2.$$
The last equation represents a circle $m_e(O_e, r_e)$ iff $r_e = \sqrt{a_1 a_2 - r_1 r_2} > 0$.
If $d = |O_1 O_2|$ then $a_1 = \dfrac{r_1}{r_2 - r_1} d$, $a_2 = \dfrac{r_2}{r_2 - r_1} d$ and holds

$$r_e = \sqrt{a_1 a_2 - r_1 r_2} = \sqrt{d^2 - \left(r_2 - r_1\right)^2}.$$

This implies that $d > r_2 - r_1$. The condition indicates that the circle k_1 intersects the circle k_2 in two points or the circle k_1 lies outside the circle k_2.

b) Without loss of generality, let us consider the point S as an internal point of the circle k_1 and an external point of the circle k_2.

Holds true that $a_1 h_2 = -a_2 h_1$ and we calculate

$$0 = a_1 h_2 + a_2 h_1 = a_1 \cdot \left(u_2^2 - r_2^2\right) + a_2 \cdot \left(u_1^2 - r_1^2\right) =$$
$$= a_1 \cdot \left[\left(x - a_2\right)^2 + y^2 - r_2^2\right] + a_2 \cdot \left[\left(x - a_1\right)^2 + y^2 - r_1^2\right].$$

It can be simplified

$$\left(x - 2\frac{a_1 a_2}{a_1 + a_2}\right)^2 + y^2 = r_1 r_2 - a_1 a_2 + 4\left(\frac{a_1 a_2}{a_1 + a_2}\right)^2.$$

This equation represents a circle m_i with a center in the point $O\left[2\dfrac{a_1 a_2}{a_1 + a_2}, 0\right]$

providing $$r_i = \sqrt{r_1 r_2 - a_1 a_2 + 4\left(\frac{a_1 a_2}{a_1 + a_2}\right)^2} > 0.$$

If a point O_i is internal center of the homothety of the circles $k_1(O_1, r_1), k_2(O_2, r_2)$ then for its signed ratio holds $(O_1 O_2 O_i) = \frac{\overrightarrow{O_1 O_i}}{\overrightarrow{O_2 O_i}} = -\frac{a_1}{a_2}$ and we derive $O_i\left[2\frac{a_1 a_2}{a_1 + a_2}, 0\right]$. The center of the circle m_i is the point O_i.

To determine the radius $r_i > 0$ of the circle m_i we obtain similarly
$$r_i = \sqrt{r_1 r_2 - a_1 a_2 + 4\left(\frac{a_1 a_2}{a_1 + a_2}\right)^2} = \ldots = \frac{\sqrt{r_1 r_2}}{r_1 + r_2}\sqrt{(r_1 + r_2)^2 - d^2}.$$
This implies that $0 < d < r_1 + r_2$. The condition means that the circles k_1, k_2 are intersecting or one circle lies inside the other.

Corollary. *The circles* $m_e(O_e, r_e)$, $m_i(O_i, r_i)$ *are circles of a pencil of circles determined with* $k_1(O_1, r_1)$, $k_2(O_2, r_2)$.

Proof. The statement follows directly from the fact that the equations of the circles $m_e(O_e, r_e)$, $m_i(O_i, r_i)$ can be written by equation $\lambda K_1 + \mu K_2 = 0$, $[\lambda, \mu] \neq [0,0]$, where
$$K_1 = (x - a_1)^2 + y^2 - r_1^2, K_2 = (x - a_2)^2 + y^2 - r_2^2.$$
If we put $[\lambda, \mu] = [1, -1]$ then we obtain $x = \frac{(a_1 + a_2)}{2a_1 a_2}(a_1 a_2 - r_1 r_2)$. This is an equation of a radical axis of the pencil of the circles $k_1(O_1, r_1)$, $k_2(O_2, r_2)$.

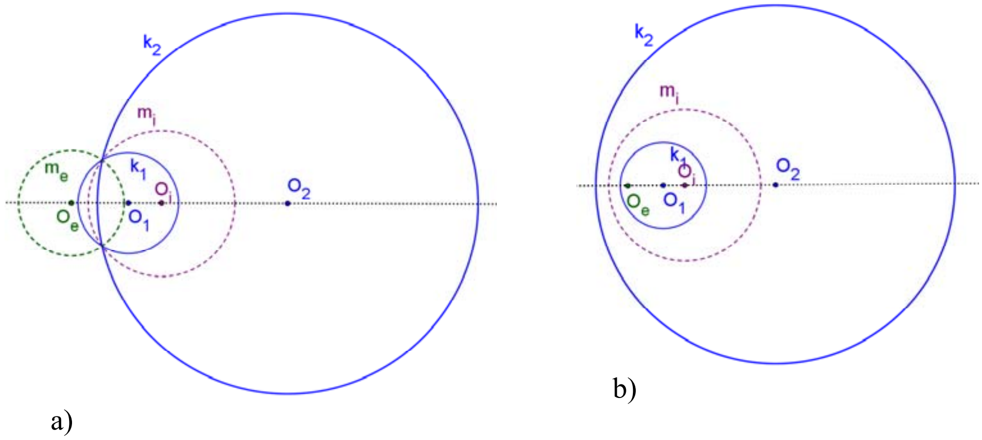

Figure 3 The pencils of the circles in different positions. The circles $m_e(O_e, r_e)$, $m_i(O_i, r_i)$ are the loci of points as centers S of circle inversion $K_i[\omega(S, r)]$ which transforms given circles $k_1(O_1, r_1)$, $k_2(O_2, r_2)$ into congruent circles. If the point $S = m_e \cap k_j, j = 1,2$ (Fig. 3a) then the inversion maps the circles k_1, k_2 into two lines. This special case we will not consider.

Outline of solution of Apollonius' problem in variant *CCC*

Let us find a solution $k_4(O_4,r_4)$ by the method of circle inversion. We will consider a general case, when given three circles k_1,k_2,k_3 have no common points and one lies outside the others. Without loss of generality assume that $r_1<r_2<r_3$, too.

A center S of circle inversion $K_i\left[\omega(S,r)\right]$ we put into an intersection of two circles m_e^{12}, m_e^{23} which are constructed according to the theorem. It holds true that $m_e^{12}\cap m_e^{23}\cap m_e^{31}=\{S_1,S_2\}$.[3] The radius r of the circle ω of the inversion K_i can be chosen arbitrary. If we put $\omega\perp k_3$ then $k_3'\equiv k_3$ and holds true that the inversion $K_i\left[\omega(S,r)\right]$ maps the circles k_1,k_2,k_3 into circles k_1',k_2',k_3 the circles $k_1',k_2',k_3'\equiv k_3$ which have the radius $r_3=r_0$.

It is evident that there exist two circles which are tangent to $k_1'(O_1',r_3),k_2'(O_2',r_3),k_3(O_3,r_3)$. The circles $k_j'(O',r_j'),\ j=4,5$ have a center O' which is a point of intersection of perpendicular bisectors $O_1'O_2',O_2'O_3$ and $O_1'O_3$ (The point O' is a power point with respect to the circles k_1',k_2',k_3, too.).

The inversion $K_i\left[\omega(S,r)\right]$ maps the circles $k_j'(O',r_j'),\ j=4,5$ into circles $k_j(O_j,r_j),\ j=4,5$ which represent two of eight possible solutions.

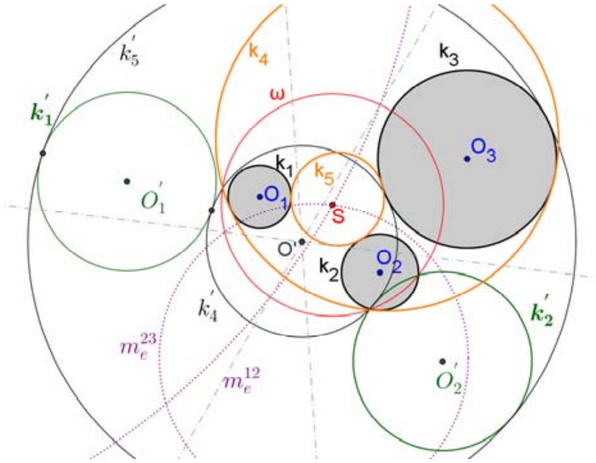

Figure 4 Two of eight possible solutions of the task.

[3] The circles $m_e^{12},m_e^{23},m_e^{31}$ belong to the same coaxal system. It follows from the corollary of the theorem and from the D'Alembert Theorem about the collinearity of centers of homotheties of three given circles. [2, p. 155] Using appropriate dynamical geometry software the reader can illustrate this fact. The rigorous proof is left to the reader as an exercise.

Let us deal with a solution in which a circle $k'(O',r')$ has tangency with the circles k_2', k_3. It is evident that the center O' is a point of the perpendicular bisector $o_{O_2'O_3}$ of the segment $O_2'O_3$ independently on the fact if there exists external or internal tangency.

Let us consider a circle $k'(O',r')$ which has an external tangency with the circle $k_1'\left(O_1',r_3\right)$ and an internal tangency with a given circle $k_2'\left(O_2',r_3\right)$ or vice – versa.

A locus of its centers O' is hyperbola h which focuses are the centers O_1', O_2'. This follows from the definition of a conic section because holds $\left\| O_1'O' \right| - \left| O_2'O' \right\| = 2r_3$. The line $o_{O_2'O_3}$ intersects hyperbola h in a center of the circle k'. There are two solutions there which the circle inversion $K_i\left[\omega(S,r)\right]$ maps into additional two solutions (Fig. 5).

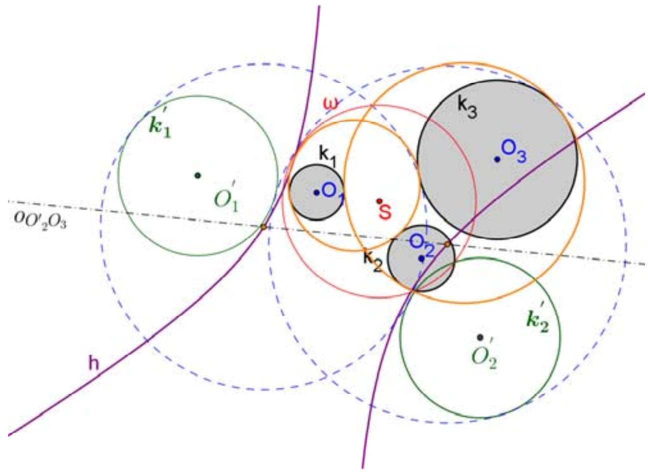

Figure 5 The additional solutions of the Apollonius' problem related to using a hyperbola.

By analogy we can draw the other four solutions if we use the perpendicular bisectors of the segments $O_1'O_3$, $O_1'O_2'$ and an appropriate hyperbola.

Conclusion

Apollonius' problem belongs to one of the most interesting geometrical tasks which solution integrated within theoretical knowledge from different domains of geometry. As we presented in the introduction, widely used is the non-linear transformation called circle inversion. In this article we apply this transformation, too.

The idea of the outlined geometrical solution is in a transition of given circles into congruent circles. As was shown two of eight solutions can be drawn immediately. To complete the solutions in details should enlarge extend of this paper significantly. This is something that can be left to the reader as a practice.

Our last note is related to a practical construction. A drawing all solutions is suitable to use an appropriate dynamic geometry software, mostly due to an interactive construction of conic sections. Some methodological approach can be found in [12, 13].

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ACTIVE METHODS OF MATHEMATICS EDUCATION

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ABSTRACT. *In the paper we deal with the survey on active methods of mathematics education. The survey was done by the research method of interview with the structure of 10 items with more possible choices and 1 item with open answer. The participants of the interview had possibility to further clarify their responses verbally to make the answers more personal. The participants were 20 teachers of selected schools in Bratislava. In the paper we describe the survey, used items, and we analyze the answers. They give us insight to the active methods used by respondents in their mathematics lessons. Survey also provided us with participants' opinions about efficacy of active methods in various years of school education, various stages of the lesson and also sources of the active methods those are these teachers using. The results of the survey showed us that participating teachers are using all sort of active methods and they are aware of the importance of such methods for mathematics education. They tend to use these methods almost in all years of education and in various areas of mathematics. Respondents are also using many sources of active methods, even creating their own.*

KEY WORDS: *active methods, mathematics education, survey*

CLASSIFICATION: *D40, C70*

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Introduction

One of the most important tasks for the teacher is to motivate the students and encourage them to play an active role in education. The teacher has a number of options to fulfil this role. One of them is the use of active teaching methods.

In the literature, the notion of active teaching methods most often denote the teaching methods, those plan, organize and manage teaching in the way that educational goals are fulfilled mostly thanks to the own cognitive activities of students [1]. Many researches support the value of active learning [2], [3], [4]. From the previous study realized on the sample of 1248 teachers in Slovakia we also know, that about 80% of them see the importance of using innovations in their teaching, what in this study include also active methods [5]. There exist very useful sources of activities those belong to the active teaching methods available for mathematics teachers in Slovakia. We can mention for example publications containing various such activities proper for the use in mathematics education [6], [7], [8], [9].

The positive results of researches and available resources encourage using of active teaching methods by mathematics teachers in Slovakia. In this article we describe survey on actual teachers'practice of selected sample of teachers in this school subject.

The aim of our survey was to determine if selected mathematics teachers in Bratislava apply active teaching methods, which of them they use, in which conditions and also to study specific active methods they apply in their mathematics lessons. Because of the small sample we cannot make any general conclusions but we can get insight to the using of active teaching methods of selected teachers and also their best practices. This insight

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and examples of best practices can contribute with valuable ideas to educators developing materials for active learning and also to other mathematics teachers in practice.

Survey sample, administration and results

Survey respondents were teachers of mathematics at primary and secondary schools (mostly grammar schools) in Bratislava. We conducted interviews with 20 teachers of mathematics, of which 6 were men and 14 were women. Of them 4 taught only in lower secondary education, while 7 at lower secondary and upper secondary education (8 years grammar school). Only in upper secondary education worked 9 respondents. Most of the respondents (16) were experienced teacher with more than 10 years of teaching experience, 4 respondents have lower than 10 years of teaching experience.

The survey was carried out in the form of the interview with the given structure of items. Before realization of the survey we made testing interview to find out if the items are proper and clear. On the basis of this we made the final version of the interview with modified items worded more clearly and accurately. We have also extended the offer of possible answers for some items. Interview that we have created comprises of 10 closed questions with multiple choice answers and one open question. For all closed questions, except the first, each respondent was allowed to choose more than one from several offered options. The items of the interview are in the table 1.

In the beginning of the interview we obtained each respondent's data needed to characterize the research sample. Then we followed with items about the use of active teaching methods in mathematics. The interviews were recorded on a digital voice recorder. Consequently, we analyzed and quantified obtained data. We tried to interpret them in the light of the opinions and attitudes expressed by teachers verbally during interviews.

The following table shows the numbers of positive answers for the items of the interview.

Q1	**Do you use active methods in mathematics education? (By active methods we mean teaching methods that somehow motivate students to be active in the lesson or they arouse their interest in mathematics.)**	
a)	yes	**19**
b)	no	**1**
In the case, that answer for Q1 was yes:		
Q2	**Which active methods you use in the education?**	
a)	didactic games and competitions	**15**
b)	interesting lectures (with mathematics theme)	**5**
c)	discussion with mathematician or with expert	**2**
d)	using interesting and actual tasks	**14**
e)	real-life based tasks using students' experience	**17**
f)	historical notes	**13**
g)	using some ICT – software, voting machines, interactive tables, applets	**15**
h)	excursion	**2**
i)	quick quiz, warm-up exercise	**11**
Q3	**Which form of the teaching you use when applying active methods?**	
a)	frontal teaching	**16**
b)	individual teaching	**7**
c)	group teaching	**17**
Q4	**In teaching of what mathematics areas you use active methods?**	
a)	all	**13**
b)	numbers, variables, computations with numbers	**3**
c)	expressions, functions, tables, diagrams	**3**
d)	geometry and measurement	**5**
e)	combinatorics, probability and statistics	**5**
f)	logic, reasoning and proofs	**1**

Q5	**In teaching of what mathematics areas you perceive your using of active methods most suitable and most effective?**	
a)	all	**4**
b)	numbers, variables, computations with numbers	**6**
c)	expressions, functions, tables, diagrams	**6**
d)	geometry and measurement	**8**
e)	combinatorics, probability and statistics	**6**
f)	logic, reasoning and proofs	**2**
Q6	**In what years of study you use active methods?**	
a)	in all years of study (lower secondary and upper secondary)	**11**
b)	in first three years of lower secondary (respondent teaches at lower secondary)	**2**
c)	in last two years of lower secondary (respondent teaches at lower secondary)	**1**
d)	in first three years of upper secondary (respondent teaches at upper secondary)	**5**
e)	in last year of upper secondary (respondent teaches at upper secondary)	**1**
Q7	**In what years of study you perceive active methods working the best?**	
a)	in all years of study (lower secondary and upper secondary)	**9**
b)	in first three years of lower secondary (respondent teaches at lower secondary)	**2**
c)	in last two years of lower secondary (respondent teaches at lower secondary)	**2**
d)	in first three years of upper secondary (respondent teaches at upper secondary)	**6**
e)	in last year of upper secondary (respondent teaches at upper secondary)	**2**
Q8	**In which phase of mathematics lesson you use mostly active methods?**	
a)	about the same in each	**11**
b)	in motivational phase	**6**
c)	in exposition phase	**4**
d)	in exercising phase	**2**
e)	in repetition phase	**2**
f)	as a homework	**0**
Q9	**In the activities those belong to active teaching methods you put attention that tasks are focused on:**	
a)	work with information	**6**
b)	that the contexts of tasks are actual and based on reality	**12**
c)	that the context is connected with scientific world	**1**
d)	work with ICT	**8**
e)	development of pupils ability to communicate mathematics concepts and to use proper mathematics language	**10**
Q10	**When planning active methods to include in your lesson you use:**	
a)	existing collections of tasks and activities	**11**
b)	online sources of activities	**14**
c)	you create activities on your own	**14**
d)	you use activities from your colleagues, those worked well for them	**8**
e)	proper popular or scientific literature	**8**
Q11	**State concrete examples of active methods those you use on your mathematics lessons and the mathematics areas where you use these activities.**	
In the case, that answer for Q1 was no:		
N1	**Why you do not use active teaching methods?**	
a)	do not have time to prepare them	**1**
b)	I think that mathematics is enough interesting on its own	**0**
c)	my pupils are enough interested in mathematics without using these methods	**1**
N2	**What activities you use in your mathematics lessons?**	
a)	from the textbooks	**1**
b)	I use tasks from different sources	**1**
c)	I use real-life and practical tasks	**1**

Table 1: Items of the interview with numbers of positive answers

Discussion

In this part of the paper we will discuss the answers to the items of the interview.

Q1: All respondents, except one, use active teaching methods. Some respondents expressed the opinion that it is one of the most essential and important tasks for the teachers. Some respondents try to incorporate at each lesson activity that is motivational and makes students active and find this as important.

One respondent who stated not using active methods is the teacher of mathematically gifted students in mathematics orientated classes, and according his opinion for activation of these students is sufficient suitable choice of mathematical tasks and problems, with emphasis on the connection with life and real examples from practice.

Q2: We can say that some active methods are used by most teachers (see results in table 1). Interesting is fact that majority of the teachers said that they use as an active method also quick quizzes or warm-up exercises. Some respondents consider these methods as warm-up and preparation for mathematics lesson and their results are not included in the student assessment. Other teachers understand them as a means of extrinsic motivation, and these quizzes are part of student assessment. Here we can see that some teachers include in notion active teaching method also the methods of assessment during that are students doing some activities and that motivates students to work harder at their lessons.

Some teachers also use other active methods, such as the creation of mathematical bulletin boards, project learning, mathematical correspondence seminars, creating conceptual maps and similar activities.

Q3: Active methods are used mostly in the form of group teaching or frontal teaching and less in the form of individual teaching. That is in accordance with typical organization of educational process during teaching methods stated in Q2.

Q4, Q5: According the answer to the Q4 the majority of the teachers try to use active methods in all areas of mathematics. The numbers of positive responses to Q5 showed that teachers see difference in efficacy of using active methods between different areas of mathematics. The most respondents think that using active methods is the most effective in the area geometry and measurement. In this area of mathematics there are many very nice sources of active teaching methods [10], [11], [12], [13], [14]. In the following areas teachers use active methods with about the same efficacy: numbers, variables, computations with numbers; expressions, functions, tables, diagrams; combinatorics, probability and statistics. These areas are ideal for the use of ICT [15].

The lowest is using of active methods in the area logic, reasoning and proofs. This fact can encourage further study on the using of active methods in this area and also development of suitable teaching materials.

Some teachers think that the highest efficiency of active methods is in the areas in which they feel most certain themselves and have the deepest knowledge of them. Several teachers see as one of the most attractive topic financial mathematics, in this topic they have success using active methods effectively.

Q6, Q7: Respondents mostly use active methods in all levels of mathematics education. As for upper secondary education, respondents are using active methods less in the last year, which is associated with the preparation for graduation and entrance exams for universities those take part in this year of education. Distribution of responses is also related to the opinions and experience of some teachers that the effectiveness of active teaching methods depends not on the particular year of education, but on a particular group of students, respectively, some teachers think it does not depend neither on that.

Q8: Teachers use active methods in various phases of the lesson, the most in the motivational phase and none of them use them in the form of the homework.

Q9: When using active methods teachers focus mostly on these things: that the contexts of tasks are actual and based on reality; the development of pupils ability to communicate mathematics concepts and to use proper mathematics language; work with ICT and with information.

Q10: Most respondents during their teaching practice formed or are forming collection of activities those belong to active teaching methods. Most often teachers create these activities themselves, or they use online sources. Some teachers use the Internet in developing of their own activities only to update information, to obtain data on the history of mathematics; others search for complete activities ready for use. Many respondents draw inspiration or complete specific active methods from existing collections of tasks and activities. Some teachers use ideas or activities provided to them by their colleagues, those have successfully tested these activities in practice. Nowadays, teachers can took ideas also in the courses and trainings of further teacher education; this way used also some of our respondents.

Q11: Teachers provide us with a wide range of different activities and methods, even whole collections of their own activities. The most frequently have been mentioned didactic games and competitions. Often they were crossword puzzles, Sudoku and Bingo, but the most common activity was Algopretek, stated by about a quarter of respondents.

Conclusion

In this paper we presented survey on using of active teaching method in mathematics education at secondary schools. The aim of our survey was to determine if selected 20 mathematics teachers in Bratislava apply active teaching methods, which of them they use, in which conditions and their opinions related to this area.

As showed our survey, the majority of respondents are aware of the importance of the using of active methods in mathematics education and they include them systematically in the educational process. This result is in accordance with previous studies those have stated interest of teachers to innovate their methods in Slovakia [5]. A positive signal is also a diversity of methods and effort to find new inspirations, whether from the Internet, existing literature, or from colleagues and also creation of their own activities.

Because of the small study sample we cannot make any general conclusions but we got insight to the using of active teaching methods of selected mathematics teachers and also their best practices. This insight can contribute with some ideas to educators developing materials for active learning and also to other mathematics teachers in practice.

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NOTES TO EXTREMA OF FUNCTIONS

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ABSTRACT. *The necessary and the sufficient condition for the existence of a local extremum give us instructions on how to find extrema and also provide a procedure to determine the kind of extremum. However, the situation may not always be exemplary, as we would expect, and for some tasks (mentioned in the paper) we also have to use other methods in order to detect local or constrained extrema.*

KEY WORDS: *function of several variables, local extrema, constrained extrema*

CLASSIFICATION: *D 50*

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Introduction

A traditional part of mathematics courses at universities is mathematical analysis. Its initial parts include also differential calculus. Students learn the concepts of derivative (one-sided, improper), differential, higher order derivatives, etc.

An important continuation of this topic is also the question of application of differential calculus, i.e. mainly tasks dealing with finding local and global extrema of functions. This area has a great use in terms of extrema of functions of one variable and extrema of functions of several variables.

Let us first mention a function of one variable. The existence of an extremum at some point is especially dealt with in Fermat's theorem: *if a function f has its local extremum at a point c and if f is differentiable at c, then* $f'(c)=0$. The character of this extreme is dealt with in the following theorem: *let c be a stationary point of a function f and let* $f''(c)\neq 0$ *; if* $f''(c)>0$ *then f has a sharp local minimum at c, and if* $f''(c)<0$ *then f has a sharp local maximum at c.*

The situation may be more complicated if we consider, for example, the function $f: y=x^{23}$, for which $f'(0)=f''(0)=...=f^{(22)}(0)=0$. However, in the end we find also a non-zero derivative of some order, and depending on its parity we decide about the (non)existence of a local extremum at a suspected stationary point.

Thus we see that even in the case of functions of one variable the situation is not always clear in advance (and we considered only cases where the derivate at the point exists), and that we do not always find a local extremum at a stationary point.

Constrained Local Extrema

Also due to the previous lines it is expected that finding extremes in case of functions of several variables does not have to be only a matter of using an algorithm. Once again, we will at first mention the necessary condition for the existence of extrema: *if a function* $f(x_1,...x_n)$ *has an extremum at a point A, and if all partial derivatives in A exist, then*

$df(A)=\sum_{i=1}^{n}\frac{\partial f(A)}{\partial x_i}dx_i=0$. Let us add a sufficient condition, which is also about the nature of local extremum (for simplicity, let us consider only a function of two variables): *let a point A be a stationary point of a function $f(x, y)$, let f be twice differentiable at A, let $D=f''_{xx}(A)f''_{yy}(A)-\left[f''_{xy}(A)\right]^2$; if $D>0$ and at the same time $f''_{xx}(A)>0$ (respectively $f''_{xx}(A)<0$), then f has a sharp local minimum (respectively maximum) at A, and if $D<0$, f has no local extremum.*

We can immediately guess the first problem – if the determinant $D=0$, we cannot determine the existence of an extremum at the stationary point.
Finding constrained extrema only adds more difficulties. In their analysis we use Lagrangian *L*. If we are looking for constrained extrema of a function $f(x, y)$ subject to the constraint $g(x, y)=0$ we create the following Lagrangian $L(x, y, \lambda)=f(x, y)+\lambda g(x, y)$.
In addition, the following theorem is valid: *if a function L has a local extremum at a point C, then function f has a constrained local extremum at C.* Unfortunately, again, this is only implication and thus we might not find all constrained extrema of *f*.
During Mathematical Analysis seminars students of course prefer standard algorithmic tasks, where one procedure leads to a correct result. We will show several tasks with "difficulties" in the solution. Either the value of the determinant *D* at stationary points is zero or even negative – i.e. Lagrangian will not have a local extremum at the given point even though the original function will have a constrained extremum at the point.

Problem 1.
Find local extrema of the function $u(x,y)=(x-y)^2+(y-1)^3$.
Solution.
Solving the system of equations $u'_x=0$, $u'_y=0$ we find the stationary point $\Xi[1;1]$. However $D(\Xi)=0$ and thus we can't decide whether the function *u* has an extremum at this point.
Let us therefore choose points Λ, Π in the neighborhood of Ξ and examine the difference $\Delta u=u(\Lambda)-u(\Xi)$, respectively $\Delta u=u(\Pi)-u(\Xi)$. Let the points Λ, Π lie on a line $y=x$, then $\Delta u=(y-1)^3$. If we select $y_\Lambda>1$, respectively $y_\Pi<1$, we get $\Delta u=u(\Lambda)-u(\Xi)>0$, respectively $\Delta u=u(\Pi)-u(\Xi)<0$. The expression Δu thus does not maintain the sign, i.e. there is no extremum at the point Ξ.

Problem 2.
Find constrained local extrema of the function $f(x,y)=xy$, if the constriction is given by the condition $x^2+y^2=2$.
Solution.

Let us create the Lagrangian $L(x,y,\lambda)=f(x,y)+\lambda g(x,y)==xy+\lambda\left(x^2+y^2-2\right)$.
Solving the system of equations $L'_x=0$, $L'_y=0$, $L'_\lambda=0$ we get the following stationary points: for the value $\lambda=-\frac{1}{2}$ points $\Gamma[1;1]$, $\Lambda[-1;-1]$; for the value $\lambda=\frac{1}{2}$ points body $\Theta[1;-1]$, $\Omega[-1;1]$.

We can easily verify that unfortunately $D(\Gamma)=D(\Lambda)=D(\Theta)=D(\Omega)=0$. The existence of extrema is therefore verified through examination of the total differential of the second order d^2L.
For the point Γ we get $d^2L(\Gamma)=-dx^2+2dx\,dy-dy^2$. Let us consider points from the neighborhood of Γ which also lie on a tangent to the circle $x^2+y^2=2$ constructed at the point Γ. Then such point $X[dx;dy]$ has to satisfy the condition $g'_x(\Gamma)dx+g'_y(\Gamma)dy=0$, i.e. $2dx+2dy=0$. Let $dy=-dx$. Then $L(X)-L(\Gamma)=dL(\Gamma)+\frac{1}{2}d^2L(\Gamma)=-2dx^2<0$, respectively $L(X)<L(\Gamma)$.
Therefore the function L has a local maximum at the point Γ, thus the function f a constrained local maximum.
Let us take the point Θ. It is obvious that $d^2L(\Theta)=dx^2+2dx\,dy+dy^2$. The points $X[dx;dy]$ from the neighborhood of Θ which also lie on a tangent to the circle $x^2+y^2=2$ constructed at the point Θ satisfying the condition $g'_x(\Theta)dx+g'_y(\Theta)dy=0$, i.e. $2dx-2dy=0$. Let $dy=dx$. Then $L(X)-L(\Theta)=dL(\Theta)+\frac{1}{2}d^2L(\Theta)=2dx^2>0$, respectively $L(X)<L(\Theta)$.

Therefore the function L has a local minimum at the point Θ, thus the function f a constrained local minimum.

Analogous analysis can show that the function f has at Λ a constrained local maximum and at the same time at Ω a constrained local minimum.

Problem 3.
Find constrained local extrema of the function $u(x,y)=x^2-y^2$, if the constriction is given by the condition $2x-y-3=0$.
Solution.

Let us define the Lagrangian $L(x,y,\lambda)=x^2-y^2+\lambda(2x-y-3)$. Solving the system of equations $L'_x=0$, $L'_y=0$, $L'_\lambda=0$ we get the stationary point $\Phi[2;1]$. However it is true that $D(\Phi)=L''_{xx}(\Phi)L''_{yy}(\Phi)-\left[L''_{xy}(\Phi)\right]^2<0$.

Thus the Lagrangian does not have a local extremum at the point. Despite this fact there really is a constrained local extremum of the function *u* at Φ. This can be verified if we express for example the variable y from the constraint and we will look for extrema of a function of one variable $u(x,2x-3)=x^2-(2x-3)^2$. Truly, for $x_0=2$ it is true that $u'(x_0)=0$ and at the same time $u''(x_0)<0$, i.e. the function $u(x,y)$ has constrained maximum at $\Phi[2;1]$.

Problem 4.

Find a cuboid with the largest volume and with a diagonal $u=2\sqrt{3}$.

Solution.

Obviously, we are trying to find the maximum of the function $V(a,b,c)=abc$, with a constraint $a^2+b^2+c^2=12$. We are looking for extrema of the Lagrangian $L(a,b,c,\lambda)=abc+\lambda\left(a^2+b^2+c^2-12\right)$. Solving the system of equations $L'_a=0$, $L'_b=0$, $L'_c=0$, $L'_\lambda=0$ we get the stationary point $\Upsilon[2;2;2]$, while $\lambda=-1$. However, this time it is true that $D(\Upsilon)=L''_{aa}(\Upsilon)L''_{bb}(\Upsilon)-\left[L''_{ab}(\Upsilon)\right]^2=0$.

The character of the extremum will be decided using the second differential, respectively the equality $L(X)-L(\Upsilon)=\frac{1}{2}d^2L(\Upsilon)$. After calculating the second order partial derivatives we obtain $L(X)-L(\Upsilon)=-(dx-dy)^2>0$, i.e. the function *f* has a local maximum at Υ. Then the "cuboid" with the largest volume satisfying the condition is in fact a cube with an edge length $a=2$.

Problem 5.

Find constrained local extrema of the function $w(x,y,z)=xyz$, if the constriction is given by the conditions $x+y-z=3$, $x-y-z=8$.

Solution.

Let us create a Lagrangian (this time of five variables) $L(x,y,z,\alpha,\beta)=xyz+\alpha(x+y-z-3)+\beta(x-y-z-8)$. Solving the system of equations

$$L'_x=0,\ L'_y=0,\ L'_z=0,\ L'_\alpha=0,\ L'_\beta=0$$

we get the stationary point $\Psi\left[\frac{11}{4};-\frac{5}{2};-\frac{11}{4}\right]$. Further calculations however show that $D(\Psi)=L''_{xx}(\Psi)L''_{yy}(\Psi)-\left[L''_{xy}(\Psi)\right]^2<0$, i.e. the Lagrangian does not have an extreme at this point.

But what about the original function w? Let us take a look at this problem once again – geometrically. Points lying in the set defined by our two constraints are actually points lying on a line which is the intersection of the two given planes. Its parametric representation is $x=t$, $y=-\frac{5}{2}$, $z=t-\frac{11}{2}$; $t\in\mathbb{R}$.

Then we can look for extrema of the function of one variable $w(t)=xyz=-\frac{5}{2}t^2+\frac{55}{4}t$ .We can easily verify that this function has a local maximum at $t_0=\frac{11}{4}$, i.e. the function $w(x,y,z)$ has a constrained local maximum at Ψ .

Conclusion

In case of both local and constrained extrema we can find – as with other topics – a number of tasks that have a demonstrational “school” character. We use a predetermined algorithm to find stationary points, and then with help of other known rules we determine the type of extremum. This article presents several non-standard tasks where, for various reasons, difficulties with the application of the said algorithm arise. Students are thus forced to meet the main objective of learning mathematics – to think.

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PROBLEMS WITH INFINITY

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ABSTRACT. *The various admittances to interpretation of infinity became the source of gravest ideological collisions and philosophical conflicts. Antinomies was adjusted in set theory by the medium of axiomatization. On the other hand there are many mathematical objects that we do not know to imagine. For example, the existence of such several objects arises from simple fact that the sets* $\mathbb{N},\mathbb{Q}$ *are equivalent. So there is a sequence of all pairwise different rational numbers, like this* $0; -1,1; -2, -\frac{1}{2}, \frac{1}{2}, 2;\ -3, -\frac{1}{3}, \frac{1}{3}, 3; -4, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{4}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4; \cdots (= \{r_n\}_{n=1}^{\infty})$ *defined through the medium the weight of a rational number. It is not possible to discover a common formula, by means of which we would say where exactly a rational number* $\frac{p}{q}$ *is found in the sequence. The function f,* $f(x) = \sum_{n:r_n \leq x} 2^{-n}$, *is defined on* $\mathbb{R}$ *correctly, but we do not know to calculate exactly its values. Furthermore, f* is increasing whereby $\mathbb{Q}$ is *the set of its points of discontinuity. It is known about the Lebesgue measure* μ *that* $\mu(\mathbb{Q}) = 0$. *Let* $\{s_n\}_{n=1}^{\infty}$ *be a sequence of all pairwise different rational numbers from the interval* $\langle 0,1 \rangle$ *and let* $J_n = \left(s_n - \frac{1}{5^n}, s_n + \frac{1}{5^n}\right)$, $n \in \mathbb{N}$. *The intervals* J_n, $n \in \mathbb{N}$, *do not cover* $\langle 0,1 \rangle$ *because* $\mu(\bigcup_{n=1}^{\infty} J_n) \leq \frac{1}{2}$*.Furthermore, the set* B, $B = \langle 0,1 \rangle - (\bigcup_{n=1}^{\infty} J_n)$, *is uncountable and* $\mu(B) \geq \frac{1}{2}$. *Nevertheless it is hard task to find anywise some element in* B.

KEY WORDS: *infinity, countable set, function, measure*

CLASSIFICATION: *C90, E60, I20, I30, I90*

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Mathematical objects with absent apprehension

We do not commit great imprecision; if we assert that whole contemporary mathematics is in practice based on set theory. This fact was occasioned by ambition to give integrating mathematics some institutional form. By one of most important intention of set theory is the effort on a comprehensive explanation of the notion infinity in mathematics. The various admittances to interpretation of infinity became the source of gravest ideological collisions and philosophical conflicts (see [3]). The fact that we explain sets as arbitrary aggregations of objects guides not only to the exceeding extension and enriching of mathematics but also to some inconveniences. These irritancies consist in overloading various domains of objects examined in traditional mathematical disciplines on new objects that we can´t to measure, but we do not know their also to conceive. We would hesitate at all to accept existence such objects before the admission of set attitude. For example, we refer the curves completely increasing a square or the continuous functions which have not derivative whether the functions which are not possible to define from principled reasons by any formula. It is noteworthy that opinions on similar demands are accustomed to change with time. Many these mathematical objects (that we considered as some pathologic counter-examples or as although entertaining but be not good for anything suitable) can fall into the cynosure on the ground of their theoretical importance otherwise their usefulness in applications. After this manner those fall in the series of "decorous" and "respectable" members of mathematical world. We assign at least one example. The attributes of fractals (geometric objects with non-integer dimension) was regarded per pathological. The fractals are the source of intensive aesthetic experiences thanks to their computerised display today. We find the ideal forms of fractals directly in the country for a wonder at each remove.

Two constructions based on countability of the set of rational numbers

In this paper all constructions make use of the fact, that it is possible to arrange all rational numbers in a sequence. It is possible for example through the medium the weight v of a rational number ($v\left(\frac{p}{q}\right) = |p| + q$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, p and q are relatively prime):

(1) $0; -1,1; -2, -\frac{1}{2}, \frac{1}{2}, 2;\ -3, -\frac{1}{3}, \frac{1}{3}, 3; -4, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{4}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4; \cdots$.

There is the most $2(v-1)$ of rational numbers with a weight v, $v > 1$. It is not possible to discover a common formula, by means of which we would say where exactly a rational number $\frac{p}{q}$ is found in the sequence (1). We say only say that $\frac{p}{q}$ is on some position among first $v(v-1)+1$ $(= 1 + 2 + 4 + \cdots + 2(v-1))$ members, where $v = |p| + q$. It´s well known that every monotone real function defined on the set $\mathbb{R}$ has only a countable set of points of discontinuity. Usually we imagine the set of points of discontinuity M of this function as a set of isolated points. For example, the set M of the function $y = [x]$equals $\mathbb{Z}$, where the symbol $[x]$denotes the whole part of the real number x. The set M can however contain as well infinitely points which are not its isolated points. The function defined on $\mathbb{R}$ by means of formula

$$f(x) = \begin{cases} k, & x = k \in \mathbb{Z} \\ k + \dfrac{1}{n+1}, & x \in \left\langle k + \frac{1}{n+1}, k + \frac{1}{n} \right), k, n \in \mathbb{N} \end{cases}$$

is non-decreasing and $M = \mathbb{Z} \cup \left\{x \in \mathbb{R}; \exists k, n \in \mathbb{N}\ x = k + \frac{1}{n+1}\right\}$. The integers are not isolated points of the set M. We can despite it advance the graph of this function although it consists of infinitely line segments and in addition of infinite set of points. The vision about this function is clear. It is hardly imagine such increasing function defined on $\mathbb{R}$ where $M = \mathbb{Q}$. However such function exists.

Task 1. Let $r_1, r_2, \cdots, r_n, \cdots$ be a sequence of all mutually different rational numbers (for example the sequence (1)). Let us define function f on the set $\mathbb{R}$ by formula

$$f(x) = \sum_{n: r_n \leq x} 2^{-n}.$$

We prove accordingly that the function f is increasing and $M = \mathbb{Q}$, where M is the set of all points discontinuity of f..

Solution. It is easy to show that $\sum_{n=1}^{\infty} a_{k_n} < \sum_{n=1}^{\infty} a_n$ for every convergent series $\sum_{n=1}^{\infty} a_n$ with positive members where $\{k_n\}_{n=1}^{\infty}$ is arbitrary increasing sequence of natural numbers which is different from the sequence $\{n\}_{n=1}^{\infty}$. Let us consider series $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} c_n$, where

$$b_n = \begin{cases} a_n, & n \notin \{k_m; m \in \mathbb{N}\} \\ 0, & n = k_m, \quad m \in \mathbb{N} \end{cases}, \quad c_n = \begin{cases} 0, & n \notin \{k_m; m \in \mathbb{N}\} \\ a_{k_m}, & n = k_m, \quad m \in \mathbb{N} \end{cases}.$$

We have $a_n = b_n + c_n$ for any $n \in \mathbb{N}$ and further

$$\sum_{n=1}^{\infty} a_{k_n} = \sum_{n=1}^{\infty} c_n < \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (b_n + c_n) = \sum_{n=1}^{\infty} a_n.$$

The previous consideration refer also to series $\sum_{n:r_n \le x_1} 2^{-n}$ $(= f(x_1))$, $\sum_{n:r_n \le x_2} 2^{-n}$ $(= f(x_2))$ for every $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. We conclude that $f(x_1) < f(x_2)$. We have shown that *f* is increasing. We prove yet that the set *M* equals $\mathbb{Q}$. Let $x \in \mathbb{Q}$, $a = r_k$. The following relationships hold for every $x \in \mathbb{R}, x < a$:

$$f(a) = 2^{-k} + \sum_{n:r_n < a} 2^{-n} > 2^{-k} + \sum_{n:r_n \le x} 2^{-n} = 2^{-k} + f(x).$$

By this means $f(x) < f(a) - 2^{-k}$ and

$$\lim_{x \to a^-} f(x) = \sup\{f(x); x < a\} \le f(a) - 2^{-k} < f(a).$$

We have proved that *f* is not continuous in the left at *a*. Furthermore $\lim_{x \to a^+} f(x) = \inf\{f(x); a < x\} \ge f(a)$. We must prove that *f* is continuous in every irrational number *b*. Let us denote s_k the sum first *k* members of series $\sum_{n:r_n < b} 2^{-n}$ $(= f(b))$. So we have $s_k = \sum_{i=1}^{k} 2^{-l_i}$, where $l_1 < l_2 < \cdots < l_k$ is *k* smallest natural numbers with property $r_{l_i} < b$, $i = 1,2,\cdots,k$. Let $r_l = \max\{r_{l_1}, r_{l_2}, \cdots, r_{l_k}\}$. Evidently $r_l < b$. Inequalities $s_k < f(r_l) < f(x) < f(b)$ are true for every $x, r_l < x < b$. So we have that

$$f(b) = \sup\{s_k; k \in \mathbb{N}\} \le \sup\{f(x); x < b\} = \lim_{x \to b^-} f(x) \le f(b)$$

and $\lim_{x \to b^-} f(x) = f(b)$. We prove also that $\lim_{x \to b^+} f(x) = f(b)$. Let $\varepsilon > 0$. Choose such $m \in \mathbb{N}$ that $\sum_{n=m+1}^{\infty} 2^{-n} < \varepsilon$. It is easy to see that there is such real number x_0, $b < x_0$, that $r_i \notin \langle b, x_0 \rangle$, $i = 1,2,\cdots,m$. We conclude that

$$\sum_{n:b < r_n \le x_0} 2^{-n} \le \sum_{n=m+1}^{\infty} 2^{-n} < \varepsilon, f(x_0) = f(b) + \sum_{n:b < r_n \le x_0} 2^{-n} < f(b) + \varepsilon,$$

$$f(b) \le \lim_{x \to b^+} f(x) = \inf\{f(x); b < x\} \le f(x_0) < f(b) + \varepsilon.$$

The equality $\lim_{x \to b^+} f(x) = f(b)$ emerges from arbitrariness positive number ε.

Remark. We do not know exactly to determine the function value $f(x)$ in any point *x*.

It is known about Lebesgue measure μ that $\mu(\mathbb{Q}) = 0$ (see [1]).

Task 2. Let $r_1, r_2, \cdots, r_n, \cdots$ be a sequence of all mutually different rational numbers from the interval $\langle 0,1 \rangle$ (for example, the sequence (1) if we "scratch out" all rational numbers that do not come under $\langle 0,1 \rangle$). Let μ be Lebesgue measure. Let $J_n = \left(r_n - \frac{1}{5^n}, r_n + \frac{1}{5^n}\right)$,

$n \in \mathbb{N}$ and let $B = \langle 0,1\rangle - (\bigcup_{n=1}^{\infty} J_n)$. Appreciate $\mu(\bigcup_{n=1}^{\infty} J_n)$, $\mu(B)$. Is possible to find some element from the set B?

Solution. Estimate

$$\mu(\bigcup_{n=1}^{\infty} J_n) \leq \sum_{n=1}^{\infty} \mu(J_n) = \sum_{n=1}^{\infty} 2\left(\tfrac{1}{5}\right)^n = \frac{\frac{2}{5}}{1-\frac{1}{5}} = \frac{1}{2},$$

$$\mu(B) \geq 1 - \mu(\bigcup_{n=1}^{\infty} J_n) \geq \frac{1}{2}.$$

Let *b* be irrational number, $b \in \langle 0,1\rangle$. We do not know to remove the existence of such r_n that $b \in \left(r_n - \frac{1}{5^n}, r_n + \frac{1}{5^n}\right)$. So we do not know to determine some concrete element in B nevertheless that the set B is uncountable and $\mu(B) \geq \frac{1}{2}$.

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DIDACTIC EMPATHY OF ELEMENTARY MATHEMATICS TEACHERS

RENÁTA ZEMANOVÁ

ABSTRACT. *This paper deals with the importance of didactic empathy of elementary mathematics teachers in connection with the current trend of constructivist way of teaching. The authors develop Hejny's method of schema-oriented education and work in the didactical environment Cube Buildings. In this environment, they assign tasks to children aged 5–6, namely to build a cube building according to a scheme and display it in any way in a plane. Before carrying out the research the authors predict children's solutions, both qualitative and quantitative. The qualitative prediction discriminates three basic conditions: correspondence with the scheme, building as a whole, and representation of the cube that is hidden in front view. Other qualitative conditions are observed with the last two conditions. The authors examine quantitative indicators in every qualitative condition defined in this way. Prediction, real data and correspondence rate are processed in tabular form. Based on the results analysis, the authors propose application in the training of future teachers of elementary mathematics.*

KEY WORDS: *didactic empathy, scheme building, cube buildings, teacher training*

CLASSIFICATION: *D49, D79*

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Problem formulation and aims of research

Current trend in the teaching of elementary mathematics in the Czech Republic is the schema-oriented education [3]. The method leans on constructivist way of the teaching process with strong emphasis on the pupil's personality, respecting it and developing it [2]. In contrast with the traditional approach to teaching, the significant amount of responsibility for finding and verifying mathematical associations (terms, claims, algorithms, etc.) is shifted from the teacher to the pupil – the pupil is the actor of the process; they are the ones who discover; the teacher coordinates and points in the right direction, presents suitable tasks. Lecturing in the traditional sense is not present. Work with pupil's errors has a significant place; it is an important diagnostic tool for the pupil's comprehension of mathematical situation, not an indicator of their failure. Emphasis is placed on peaceful and friendly teaching environment with mutual trust (between the teacher and the pupil), which is a necessary condition for pupils' discoveries. For this reason, we examine the ability of the teacher to sympathise with the pupils' feelings and thinking in order to optimize their own thinking and acting to a) respect the pupil's personality in its feelings (for good mood in the classroom) and b) develop pupil's personality in thinking (to arrive at mathematical discoveries).

By discovery we understand the shift from "don't know, don't understand, can't" to "know, understand and can", in which the pupil is the main actor. Thus it is a "discovery" on individual level, in connection with the specific pupil. The discovery is accompanied with the feeling of happiness – "the eureka effect", which serves as powerful motivation for further attempts at discoveries. We support that significant pupils' discoveries (first pupil in the class, unconventional solution…) be presented to other pupils, who can then use them for their own discoveries. The important aspect is that it is not the teacher who presents the "discoveries".

All the above mentioned puts increased requirements on the teacher. First of all they must accept their changed pedagogical role, whether they are an active teacher with experience or a pedagogy student, who remembers the traditional teacher's role from their own school years.

The aim of our long term research is to develop and verify tools to support the development of didactic empathy of teachers of elementary mathematics. We work in various didactical environments and utilize different methods. Currently, we have chosen the didactical environment of cube buildings and method of comparing the records. We base our work on the theoretical notion of procept by Gray and Tall, who introduced this term for mathematical objects that can be perceived to signify process and concept at the same time [1]. We expand the theory by amalgam of Hejny, who applies their ideas to geometric environment [3]. We perceive the final structure as concept, its creation as process, and the duality of the concept and the process of building as amalgam. In experiments, we strive to use the amalgam transfer, i.e. the transfer of conceptually described structure to the process and the other way round. We take into account the research results of Krpec in mental schemes of children [4].

Methodology

106 pupils aged 5–6 took part in the experiment in two phases: on 6. 2. 2013 it was 21 children from kindergartens from Hlubočec, Pustá Polom and Budišovice; and between 16.–26. 1. 2014 it was 85 children from various kindergartens in the Moravian-Silesian region (total of 20 schools). We worked with each child individually. We placed a cube building in front of the child without their seeing the building process, Fig. 1.

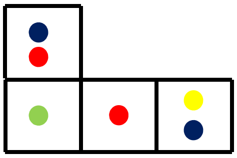

Figure 1: Given cube building

Each child had six coloured cubes at their disposal (see Fig. 1), four colour crayons corresponding with the colours of the cubes and a sheet of A4 sized paper. In the first part of the experiment we asked the child to build the same building they saw. This task was included for the child to experience the amalgam transfer: input concept (provided building) – process (building by a scheme) – output concept (their own building). In the second part of the experiment we asked the child to use crayons to draw the building in such way that somebody else could build it the same way according to their drawing.

Before this experiment we had created a prediction of children's inputs in which we recorded individual ways of solving both parts of the experiment and their percentage proportions

The presentation of the results and their analysis

In predicting the first part of the experiment (building by a scheme) we anticipated that all children will build the building correctly. In predicting the second part of the experiment we anticipated that 90% of the children will have three squares next to each other in corresponding colours (from left: green, red & blue), but will differ in representing the other cubes, 5% will use top view (cubes on top of each other will be represented by using two corresponding colours in one square or symbols will be used – lines, pips, smaller squares, etc.) and 5% of the children will not start to solve the task.

As far as the representation of the yellow cube in the first group of children is concerned, from the 90% of children, 90% represents the yellow cube in front view, 10% in top view (draw the yellow colour over the blue one or otherwise mark two colours into one square). Representing the "hidden" red cube in the same group, 70% of the children mark it in top view (Fig. 2a, b, c), 20% do not represent it at all (Fig. 3) and 10% of the children represent the red cube as square (not in front view or top view), Fig. 4. The blue cube is represented depending on the red cube representation either as square "above" its image (20%) or either in front view or top view – this cannot be determined, but we assume that those who do not use top view for the red cube, will not use it for the blue one either, i.e. perceives the representation as front view (80%).

Figures 2a, b, c: Hidden red cube in top view

Figure 3: Hidden red cube is missing

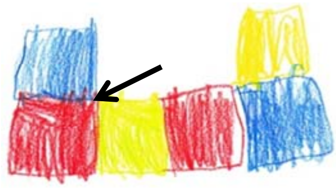

Figure 4: Hidden red cube represented differently

We processed the overview of predictions and results into tables. The **predictions** are presented in both percentages (columns with % headings) and absolute numbers (columns with No. headings), except Table 3 with absolute numbers only; **results** in absolute numbers (columns with No. headings) and the difference between predictions and results in absolute numbers is put on a scale (++ for 1–4 number difference, + for 5–9 number difference, +− for 10–14 number difference, − for 15–19 number difference and −− for 20–30 number difference). The prediction numbers were set using qualified guess. In Table 1 we compare the prediction and results of building by scheme: 13 children did not build the building correctly, the difference between the prediction and results is significant in its quality (the occurrence of incorrect building was not expected at all). Table 2 compares the prediction and results of representing the building as a whole; we consider the difference in the "Not Solving" category significant (all children tried to come up with a solution). Table 3 shows the overview of the predictions and results of representing the yellow, "hidden" red and blue cube in the 2nd upper plane. In this case, we only did a good prediction of the yellow cube representation; with the representation of the red cube we did not expect any other form than square (other image came up 16 times), We expected the children not to represent the cube at all (only 6 occurrences); to put it in front view, or top view (only 48 occurrences); and not to use a square outside the given front view, or top view (31 occurrences). It must be noted that we considered the representation of this cube to be particularly difficult. The representation of the blue cube was dependent on the representation of the “hidden” red cube, but with the exception of top view representation we predicted the numbers better.

The table does not show some specific and interesting solutions of the children, but given the focus of this research, we do not include them here.

The key aim of our research was to do an analysis of expected task solutions (qualitatively: not to omit any solution; quantitatively: to maximally approximate to real results). The key result of our research was the comparison of the prediction with real-life results. We did not omit any solution in the qualitative part; therefore the experiment can be evaluated in the given parameters. In the quantitative part we predicted some parameters wrong to a high degree (--) or wrong (-).

Building is identical				Building is not identical			
prediction		result		prediction		result	
%	No.	No.	Evaluation	%	Evaluation	No.	Evaluation
100	106	93	0	0	0	13	0

Table 1: Prediction – building by the scheme

Three squares next to each other, differences in further representation				Top view with more colours in one square				Not solving			
prediction		result		prediction		result		prediction		result	
%	No.	No.	Evaluation	%	No.	No.	Evaluation	%	No.	No.	Evaluation
90	96	101	+	5	5	5	++	5	5	0	+

Table 2: Prediction – representing the whole building

	Front view (F)			Top view (T)			Square outside of F and T			Not represented			Rectangle or otherwise		
cube	pred.	result		pred.	result		pred	result		pred.	result		pred.	result	
	No.	No.	Eval.	No.	No.	Eval.	No.	No.	Eval.	No.	No.	Eval.	No.	No.	Eval.
yellow	91	95	++	10	2	+	0	3	++	0	1	++	0	0	++
red	0	5	+	71	43	−−	10	31	−−	20	6	0	0	16	−
blue	81	85	++	20	4	−	0	7	+	0	3	++	0	2	++

Table 3: Prediction of representing the yellow cube, “hidden” red one and the blue one on top of the red one

Conclusion

Providing and evaluating the prediction in both components (qualitative and quantitative) is a significant process in bolstering didactic competencies of teachers regardless of their success rate (including significant differences between the predictions and results). With a higher number of predicted tasks and reflections on success rates, we expect the improvement of didactic empathy of teachers. This should firstly lead to improved predictions of task solving (more detailed and precise with smaller differences between the predictions and results) and secondly help with complex didactic work (communication, formulating questions, creation and grading of exercises, evaluating solutions…). This would be the subject of further research.

We intend to use this experiment to develop prediction abilities of future teachers of elementary mathematics, who would try to do their own predictions of children's results in the first phase, in the same way we did, and add percentage values of predictions in given categories in the second phase. These categories will be created on the basis of the results of the hereby presented research. The result would be a differentiation set of predictions and results of children for each student, which could be further expanded on.

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