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ON A CERTAIN GROUP OF LINEAR SECOND-ORDER DIFFERENTIAL OPERATORS OF THE HILL-TYPE

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ABSTRACT. *The Hill differential equation often occurs in physical, technical and astronomical topics, in particular in modelling connected with vibrations of mechanical systems. It is a special equation in Jacobi form investigations of which is motivated by concrete modelling time functions. In the contribution there is also established isomorphism between a certain group of Hill type differential operators and the subgroup of the group of third order differential operators with constant coefficients.*

KEY WORDS: *group, binary operation, differential equation, differential operator*

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The application of algebraic methods in the theory of linear differential equations offers new views on the qualitative theory of differential equations and provides to reveal new interesting connections which can lead to new valuable results. As it was shown in appropriate literature [3, 11, 12], in the 1950s professor Otakar Borůvka started a systematic study of global properties of linear differential second-order equations. Following Borůvka´s approach, František Neuman obtained further significant results, which generalized previously acquired results. This topic was also dealt with by other members of Borůvka´s differential equations seminar. For the constituting scientific branch there is characteristic the application of not only analytical methods used while examining differential equations, but also the usage of algebraic, topological and geometrical tools. Local methods and results are not sufficient when problems of global nature are studied, e.g. boundedness, periodicity, asymptotic and oscillatory behaviour of equations solutions, factorization of operators formed of left sides, and other properties.

In this article, which follows up the topic of a series of works devoted to algebraic properties of linear differential operators formed of left sides of linear differential equations, including the properties of groups and hypergroups of their solution spaces, we deal with the hypergroup of operators formed of operators – the left sides of second-order linear differential equations of the Hill-type. These are equations within the Jacobi form, specially equations with periodic coefficients. The Hill equation is of the form

$$y'' + \left[\Phi(x) + \lambda\right] \cdot y = 0 \ , \qquad \text{(H)}$$

with periodic function $\Phi(x)$. The differential equation of this type often occurs in physical, technical and astronomical topics, especially while solving problems connected with vibrations of mechanical systems.

In monography [8], p. 411 there is contained the Hill equation (as example 2.20) in the form

$$y'' + (ae^{2x} + be^{x} + c)\, y = 0 \qquad (1)$$

with the periodic function $\Phi(x) = ae^{2x} + be^{x}$ with the complex period $2\pi i$. After the transformation $y(x) = u(t),\ t = ix,$ we obtain the above equation (H). Indeed for $y(x) = u(ix),\ t = ix$ we have

$$u''(t) - (ae^{-2ti} + be^{-ti} + c)u(t) = 0,$$

thus
$$u(t) + \left[\Phi(t) + \lambda\right]\cdot u(t) = 0,$$

where $\Phi(t) = -ae^{-2ti} + be^{-ti} = -a\cos 2t - b\cos t + (a\sin 2t + b\sin t)\,i$, which is a periodic function with the basic period 2π and $\lambda = -c$.

Using the substitution $y = u(t)e^{-\frac{1}{2}x}$, $t = e^{x}$ we obtain from (1) the equation

$$t^2u''(t) + (at^2 + bt + c + \frac{1}{4})u(t) = 0, \qquad (2)$$

(cf. [8] p. 411), which is a differential second-order equation with polynomial coefficients. Moreover the above presented equations (1) and (2) are linear differential second-order equations in so called Jacobi form. It has been mentioned in [1, 6], Otakar Borůvka has obtained a criterion of a global equivalence for the second-order differential equations within the Jacobi form and he also found corresponding global canonical forms for such equations. For more information see F. Neuman [11, 12]. Notice, that second-order linear differential equations we also obtain as equations of functions modelling certain time processes. For example, consider the functions of the Gaussian-shaped pulse signal $v(t) = a\exp(-2\pi t^2),\ t \in \langle 0,\infty)$. Corresponding differential equation has the form

$$v''(t) - 16a\pi t^2 v(t) = 0,\quad t \in \langle 0,\infty). \qquad (3)$$

with initial conditions $v(0) = a,\ v'(0) = 0$.

For the shape analysis of non-periodical time signals (or impulses determined by radiation) there is used in [7] product of simple quadratic or cubic polynomials with exponential functions. There are functions

$$\psi(t) = at^2\exp(-\lambda t),\ \varphi(t) = at^3\exp(-\lambda t),\quad t \in \langle 0,\infty).$$

Considering the general modelling time function

$$\varphi(t) = t^n\exp(-\lambda t),\ n = 2,3,\dots,\quad t \in \langle 0,\infty)$$

we obtain the second-order equation in the Jacobi form

$$\varphi''(t) + p(t)\varphi(t) = 0, \qquad (4)$$

where $p(t) = (-\lambda^2t^2 + 2\lambda nt + n(n-1))\,t^{n-2},\ t \in \langle 1,\infty)$

with initial conditions $\varphi(1) = \exp(-\lambda),\ \varphi'(1) = (n-\lambda)\exp(-\lambda)$. The equation (4) can be rewritten into the form

$$t^{n-2}\varphi''(t) - (\lambda^2t^2 - 2\lambda nt + n(n-1))\,\varphi(t) = 0,$$

which is a second-order linear differential equation with polynomial coefficients.

Finally, the modelling time function $y(t) = A(1-\exp(-ct))^b$ called the Chapman-Richardson`s function, which is one of the most common functions based on the original Bertalanffy equation derived for growth and increment of body weight, leads also to second-order differential equation of the Jacobi form – cf. [9, 10].

Algebraic properties of structures formed by ordinary differential operators of the second order and n-th order as well (formed of left hand sides of corresponding homogeneous differential equations) have been studied in several papers. Let us mention at least [1, 2, 4, 5, 6, 9, 10, 12].

Suppose, $p,\ q: I \to \mathbf{R}$ are continuous functions,

$$L(p, q)\, y = y'' + p(x)\, y' + q(x)\, y,\ x \in I \subseteq \mathbf{R},\ y \in \mathbf{C}^2(I)$$

is a second order differential operator. Denoting

$$\mathbf{LA}_2(I)_q = \{L(p, q);\ p, q \in \mathbf{C}(I),\ q(x) \neq 0\},$$

$$\mathbf{JA}_2(I)_q = \{L(0, q) \in \mathbf{LA}_2(I)\ ;\ q \in \mathbf{C}(I),\ q(x) \neq 0\},$$

we have according to [1] (or theorem 10, [6]) that for the binary operation

$$\bullet_B : \mathbf{LA}_2(I)_q \times \mathbf{LA}_2(I)_q \to \mathbf{LA}_2(I)_q$$

defined by $$L(p_1, q_1) \bullet_B L(p_2, q_2) = L(p_1 q_2 + p_2,\ q_1 q_2)$$

the groupoid $(\mathbf{LA}_2(I)_q, \bullet_B)$ is a non-commutative group with the unit $L(0,1)$ assigning to any function $f \in \mathbf{C}^2(I)$ the function $f'' + f$. Further, if we denote

$$\mathbf{J}_C\mathbf{A}_2(I)_q = \{L(0, r)\ ;\ r \in \mathbf{R},\ r \neq 0\}$$

then with respect to Theorem 2 [6] we have that the subgroupoid $(\mathbf{J}_C\mathbf{A}_2(I)_q, \bullet_B)$ of the group $(\mathbf{JA}_2(I)_q, \bullet_B)$ is its normal commutative subgroup. Other details can be found in the paper [6].

For any triad of real or complex numbers $a, b, c \in \mathbf{R}\ (\mathbf{C})$ define

$$L(0, (a, b, c))\, y = y'' + (ae^{2x} + be^{x} + c)\, y,\ y \in \mathbf{C}^2(I)$$

and put $\mathbf{H}_2(I) = \{L(0, (a, b, c));\ a, b, c \in \mathbf{R}\ (\mathbf{C}),\ c \neq 0\}$. There is possible to define various binary operations on the set $\mathbf{H}_2(I)$. The following possibility is in connection with linear differential operators of the third order – [2, 5].

Suppose $L(0, (a_0, a_1, a_2)), L(0, (b_0, b_1, b_2)) \in \mathbf{H}_2(I)$, with $a_2 \neq 0 \neq b_2$. Define

$$L(0, (a_0, a_1, a_2)) \circ L(0, (b_0, b_1, b_2)) = L(0, (a_2 b_0 + a_0,\ a_2 b_1 + a_1,\ a_2 b_2)).$$

It can be easily verified that the above operation creates on $\mathbf{H}_2(I)$ a structure of non-commutative group with the unit $L(0, (0,0,1))$. If $L(0, (a, b, c)) \in \mathbf{H}_2(I)$, then the inverse element to this operation is $L^{-1}(0, (a,b,c)) = L\left(0, \left(-\frac{a}{c}, -\frac{b}{c}, \frac{1}{c}\right)\right)$. Indeed

$$L(0,(a,b,c))\circ L^{-1}(0,(a,b,c)) = L(0,(a,b,c))\circ L\left(0,\left(-\frac{a}{c},-\frac{b}{c},\frac{1}{c}\right)\right)=$$

$$L\left(0,\left(-\frac{ca}{c}+a,-\frac{cb}{c}+b,\frac{c}{c}\right)\right)=L(0,\ (0,0,1)).$$

The group $(\mathbf{H}_2\,(I),\ \circ)$ is isomorphic to the group $(\mathbf{L}_C\,\mathbf{A}_3(I),\circ_2)$ of linear third-order differential operators with constant coefficients. These operators are of the form

$$L(p_0,\,p_1,\,p_2)\,y(x) = y'''(x)+\sum_{k=0}^{2}p_k y^{(k)}(x)\,.$$

The group $(\mathbf{L}_C\,\mathbf{A}_3(I),\circ_2)$ is a subgroup of the group $(\mathbf{L}\,\mathbf{A}_3(I),\circ_2)$ which is investigated in [2, 5]. The corresponding isomorphism $F : (\mathbf{L}_C\mathbf{A}_3(I),\circ) \to (\mathbf{H}_2\,(I),\circ)$ is defined by

$F(L(p_0,\,p_1,\,p_2)) = L(0,\ (p_0,\,p_1,\,p_2))$ for any triad $[p_0,\,p_1,\,p_2]\ \in \mathbf{R}\times\mathbf{R}\times(\mathbf{R}-\{0\})$.

In fact to each third-order differential operator

$$L(p_0,\,p_1,\,p_2) = \frac{d^3}{dx^3}+p_0\frac{d^2}{dx^2}+p_1\frac{d}{dx}+p_2 Id$$

there is assigned the operator

$$L(0,\ (p_0,\,p_1,\,p_2)) = \frac{d^2}{dx^2}+(p_0e^{2x}+p_1e^{x}+p_2)Id$$

of the Hill-type with $p_k\in\mathbf{R}$, $p_2\neq 0$.

The simplest binary operation on $\mathbf{H}_2\,(I)$ is motivated by direct products of triads of real numbers which leads-after application of so called Ends-lemma-[13] to binary hyperstructures. Considerations of this direction deserve to be investigated in a separate paper.

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