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FOURIER TRANSFORM AND ITS APPLICATION

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ABSTRACT. *By means of Fourier series can be described various examples of wave motion, such as the sound, or wave of the earthquake. It can be used in many research or work, such as the data analysis after an earthquake or digitizing music. Generalization of Fourier series, which allows for some applications more appropriate expression is the Fourier integral. Fourier transform based on a Fourier integral in the complex form. Fourier transform is an important tool in a number of scientific fields. Its advantages, disadvantages and subtleties have been examined many times by dozens of mathematicians, physicists and engineers. In this contribution we try to summarize important aspects of this transform and discuss variety of its uses in contemporary science with emphasis on demonstrating connections to dynamic interactions in the vehicle-roadway system.*

KEY WORDS: *Fourier transform, time series, frequency representation.*

CLASSIFICATION: *I55, I85, M55*

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1 Introduction

Empirical measurements in various domains – economical, technical, or other – are often turned into time series. Based on this it is possible to perform analysis, which in turn allows us to better understand the dynamics of the factors involved. To this end, we use the Fourier transform. Being discovered at the turn of the 19th century, the theory of the Fourier transform is currently used in signal processing such as in image sharpening, noise filtering, etc. For us the relevant application is in the theory of dynamic interaction in the vehicle-roadway system.

The basis of the Fourier transform is the so-called Fourier mapping, i.e., the transformation of one function to another, from properties of which we can obtain information about the original function. Fourier transform expresses a time-dependent signal using harmonic signals, i.e., the sine and cosine functions, in general functions of complex exponentials. It is used to transform signals from the time domain to the frequency domain. A signal can be continuous or discrete. See Figure 1 for a depiction of the correspondence between the time-based and frequency-based representation.

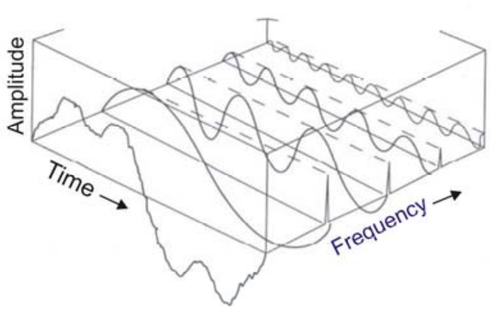

Figure 1: Amplitude frequency diagram

2 Fourier integral

A generalization of the Fourier series that permits in some applications a more appropriate expression of a non-periodical function defined almost everywhere in R is the Fourier integral.

Theorem 1: Let $f\colon R\to R$ be a function that

a) is piece-wise continuous on R along with its derivative f',

b) is absolutely integrable on R, i.e., $\int_{-\infty}^{\infty}|f(t)|\,\mathrm{d}t$ converges.

Then all $t\in R$ satisfy

$$\tilde{f}(t)=\frac{1}{\pi}\int_0^\infty \mathrm{d}\omega\int_{-\infty}^{\infty} f(s)\cos\omega(t-s)\mathrm{d}s\,,\text{ where }\tilde{f}(t)=\frac{1}{2}\left[\lim_{s\to t+}f(s)+\lim_{s\to t-}f(s)\right]. \tag{1}$$

Note 1: From the claim of Theorem 1, it follows that the values of the double integral on the right-hand side of (Eq. 1) is equal to $f(t)$ for each $t\in R$ in which f is continuous and is equal to the arithmetic mean of the left- and right- limits of this function in each point of discontinuity, provided the conditions a) and b) of this claim hold.

Note 2: Using the fact that $\forall t\in R$, $\forall s\in R$ and $\forall\omega\in\langle 0;\infty)$ we have $\cos\omega(t-s)=$ $=\cos\omega t\cos\omega s+\sin\omega t\sin\omega s$, we can rewrite (1) in the form

$$\tilde{f}(t)=\int_0^\infty[a(\omega)\cos\omega t+b(\omega)\sin\omega t]\mathrm{d}\omega\,, \tag{2}$$

where $a(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(s)\cos\omega s\,\mathrm{d}s$, $b(\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}f(s)\sin\omega s\,\mathrm{d}s$, $\omega\in\langle 0;\infty)$. (3)

Definition 1: The right-hand side of (Eq. 1) is called the *double Fourier integral* of $f\colon R\to R$. The right-hand side of (Eq. 2) is the *single Fourier integral* of f.

Note 3: It is not difficult to see that the single Fourier integral (Eq. 2) is a generalization of the double Fourier integral and the functions $a\colon\langle 0;\infty)\to R$, $b\colon\langle 0;\infty)\to R$ defined by (Eq. 3) are a generalization of the standard Fourier coefficients of a periodic function. It is clear that if $f\colon R\to R$ is an even function, then $b(\omega)=0$, $a(\omega)=\frac{2}{\pi}\int_0^\infty f(s)\cos\omega s\,\mathrm{d}s$ and $\tilde{f}(t)=\int_0^\infty a(\omega)\cos\omega t\,\mathrm{d}\omega=\frac{2}{\pi}\int_0^\infty\mathrm{d}\omega\int_0^\infty f(s)\cos\omega s\cos\omega t\,\mathrm{d}s$. Similarly, if f is odd, we have: $a(\omega)=0$, $b(\omega)=\frac{2}{\pi}\int_0^\infty f(s)\sin\omega s\,\mathrm{d}s$ and we have $\tilde{f}(t)=\int_0^\infty b(\omega)\sin\omega t\,\mathrm{d}\omega=\frac{2}{\pi}\int_0^\infty d\omega\int_0^\infty f(s)\sin\omega s\sin\omega t\,\mathrm{d}s$.

Note 4: Using the well-known Euler's formula for exponential and goniometric functions: $\cos\omega t=\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\omega t}+\mathrm{e}^{-\mathrm{i}\omega t}\right)$, $\sin\omega t=\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\omega t}-\mathrm{e}^{-\mathrm{i}\omega t}\right)$ in the single Fourier integral (Eq. 2), we obtain $\forall t\in R$: $\tilde{f}(t)=\int_0^\infty\left[\frac{a(\omega)-\mathrm{i}b(\omega)}{2}\mathrm{e}^{\mathrm{i}\omega t}+\frac{a(\omega)+\mathrm{i}b(\omega)}{2}\mathrm{e}^{-\mathrm{i}\omega t}\right]\mathrm{d}\omega$.

By letting $\frac{a(\omega)-\mathrm{i}b(\omega)}{2}=c(\omega)$, $\frac{a(\omega)+\mathrm{i}b(\omega)}{2}=c(-\omega)=\overline{c(\omega)}$ for $\omega\in\langle 0;\infty)$, we have for all $t\in R$: $\tilde{f}(t)=\int_{-\infty}^{\infty}c(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$, (4)

where $c(\omega)=\frac{1}{2}[a(\omega)-\mathrm{i}b(\omega)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(s)\mathrm{e}^{-\mathrm{i}\omega s}\mathrm{d}s$ for all $\omega\in R$. (5)

Definition 2: The right-hand side of (4), where $c\colon R\to C$ is defined by (5) is called the *Fourier integral* of $f\colon R\to R$ in a *complex form*.

3 Fourier transform

Definition 3: Let f: $R \to R$ along with its derivative f' be piece-wise continuous on R, and let f be absolutely integrable on R. Then we call f the *source of Fourier transform*. Let the set of such functions f: $R \to R$ be denoted by D_F. Then the function F: $\mathrm{i}\omega \to F(f(t))$,

where $F(f(t)) = F(\mathrm{i}\omega) = \int_{-\infty}^{\infty} f(t)\,\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$, (6)

where $\omega \in (-\infty;\infty)$ is called the *Fourier image* of f and the mapping F from the set of functions D_F defined by (Eq. 6) is called the *forward Fourier transform*.

Theorem 2: If $f \in D_F$, then there exists a Fourier image $F(i\omega)$ of f defined by (Eq.6), which satisfies $\tilde{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$, (7)

where $\tilde{f}(t) = \frac{1}{2}\left[\lim_{s\to t+} f(s) + \lim_{s\to t-} f(s)\right]$ for all $t \in R$.

Definition 4: The mapping $F^{-1}(D_F)$ defined by (7) is called the *inverse Fourier transform*, i.e., $F^{-1}(F(\mathrm{i}\omega)) = \tilde{f}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mathrm{i}\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$, $t \in R$.

Note 5: The Fourier image $F(\mathrm{i}\omega)$ is in the technical literature often called the *spectral characteristic* of f. Its magnitude $F(\omega) = |F(\mathrm{i}\omega)|$ is the *amplitude characteristic* of f, function $\alpha(\omega) = -\operatorname{Arg} F(\mathrm{i}\omega)$, $\omega \in (-\infty;\infty)$ is called the *phase characteristic* of f and the function $P(\omega) = |F(\mathrm{i}\omega)|^2$ *power characteristic* (*power spectrum*) of f. Hence, for all $\omega \in R$

$F(i\omega) = F(\omega)\,\mathrm{e}^{-\mathrm{i}\alpha(\omega)} = A(\omega) - \mathrm{i}B(\omega)$, where $A(\omega) = \int_{-\infty}^{\infty} f(t)\cos\omega t\,\mathrm{d}t$, $B(\omega) = \int_{-\infty}^{\infty} f(t)\sin\omega t\,\mathrm{d}t$

.

From this it follows that $F(\omega) = \sqrt{A^2(\omega) + B^2(\omega)}$, $\alpha(\omega) = \arctan[B(\omega)/A(\omega)]$, which means that the amplitude function $F(\omega)$ is an even function and the phase function $\alpha(\omega)$ is an odd function of the independent variable (frequency) ω.

4 Use of Fourier transform in solving representative problems

Recall that according to the Euler's formula we can write $\mathrm{e}^{\mathrm{i}\omega t} = \cos\omega t + \mathrm{i}\sin\omega t$.

Example 1: Find the Fourier image of the function $f(t)$: $R \to R$, $f(t) = \mathrm{e}^{-a|t|}$, $a \in R^+$.

Solution: From equations (Eq. 4) and (Eq. 5) it follows that

$$c(\omega) = \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-a|s|}\,\mathrm{e}^{-\mathrm{i}\omega s}\mathrm{d}s = \tfrac{1}{2\pi}\left(\int_{-\infty}^{0} \mathrm{e}^{(a-\mathrm{i}\omega)s}\mathrm{d}s + \int_{0}^{\infty} \mathrm{e}^{-(a+\mathrm{i}\omega)s}\mathrm{d}s\right) = \tfrac{1}{2\pi}\left(\left[\frac{\mathrm{e}^{(a-\mathrm{i}\omega)s}}{a-\mathrm{i}\omega}\right]_{-\infty}^{0} - \left[\frac{\mathrm{e}^{-(a+\mathrm{i}\omega)s}}{a+\mathrm{i}\omega}\right]_{0}^{\infty}\right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{a-\mathrm{i}\omega} + \frac{1}{a+\mathrm{i}\omega}\right) = \frac{1}{\pi}\frac{1}{a+\omega^2}, \text{ i.e., } \tilde{f}(t) = f(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\omega t}}{a+\omega^2}\mathrm{d}\omega.$$

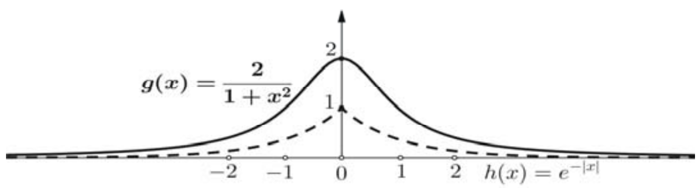

Figure 2: Comparing a graph of a function with its Fourier image

Therefore the Fourier image of $\mathrm{e}^{-a|t|}$ is $F(\mathrm{e}^{-a|t|}) = \frac{1}{a}\cdot\frac{2}{1+\left(\frac{\omega}{a}\right)^2} = \frac{2}{a^2+\omega^2}$.

Fourier transform has a wide variety of uses; we have already shown some of them for illustration. Fourier transform is also used to solve differential equations. The key idea is that the Fourier transform transforms the operation of taking derivatives into multiplication of the image by the independent variable. If we perform the Fourier transform using all independent variables we obtain as image a solution of the equation with no derivatives. When we solve it, it suffices to find the Fourier preimage which usually is the most difficult part. Unfortunately, it can also happen that the solution has no preimage. Then this method does not work. However, we may perform the Fourier transform using only some independent variables. This yields a differential equation with fewer variables and with parameters, which might be easier to solve than the original equation; nonetheless the ultimate difficulty may still be in finding the preimage.

Example 2: Using the Fourier transform find a solution of the differential equation satisfying the following conditions:

a) $y'(t) + k\,y\,(t) = a\,e^{-|t|}$, where $k \in R^+ - \{1\}$, $b \in R$, $t \in R$, $\lim\limits_{t\to-\infty} y(t) = \lim\limits_{t\to\infty} y(t) = 0$,

b) $y''(t) + 3y'(t) + 2\,y\,(t) = e^{-|t|}$, $t \in R$, $\lim\limits_{t\to-\infty} y(t) = \lim\limits_{t\to\infty} y(t) = 0$.

Note: Fourier transform can also be used for solving ordinary linear differential equations with constant coefficients assuming that the solution of such equation along with its derivatives of order up to the order of the equation has properties from Definition 3.

Solution: Let y, y', $y'' \in D_F$ and write $F(y(t)) = Y(i\omega)$. Then $F(y'(t)) = i\omega Y(i\omega)$, $F(y''(t)) = -\omega^2 Y(i\omega)$.

a) Since $F(a\,e^{-|t|}) = a\frac{2}{1+\omega^2}$, we have $y(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} Y(i\omega)e^{i\omega t}dt = \frac{a}{\pi}\int\limits_{-\infty}^{\infty}\frac{1}{1+\omega^2}e^{i\omega t}dt$.

Thus $y(t) = \frac{ae^t}{k+1}$ for $t \in (-\infty;0)$, $y(t) = a\left(\frac{e^{-t}}{k-1} - 2\frac{e^{-kt}}{k^2-1}\right)$ for $t \in \langle 0;\infty)$.

b) Since $F(e^{-|t|}) = \frac{2}{1+\omega^2}$, it follows that $y(t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} Y(i\omega)e^{i\omega t}dt = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\frac{1}{1+\omega^2}e^{i\omega t}dt$.

Hence $y(t) = \frac{1}{6}e^t$ for $t \in (-\infty;0)$, $y(0) = \frac{1}{6}$, and $y(t) = \frac{2t-1}{2}e^{-t} + \frac{2}{3}e^{-2t}$ for $t \in (0;\infty)$.

Example 3: Some situations require specifying the problem using a diagram. Here the subject of analysis is the so-called *quarter model* of a vehicle shown in Figure 3. This computational model represents one half of one axle of a vehicle. Unevenness of the road surface is the main source of kinematic excitation of the vehicle. The vehicle's response to this excitation can be found numerically in both the time and frequency domain. In the time domain we are mainly interested in time evolution of contact forces and in the frequency domain in the power spectral densities of power forces in relation to the power spectral densities of the unevenness of the road [1].

As an example, we use numerical characteristics of the vehicle Tatra model T148.

Weight parameters of the model: $m_1 = 2\,930$ kg $\quad m_2 = 455$ kg

Rigidity constants of the coupling: $k_1 = 143\,716{,}5\ \mathrm{N\cdot m^{-1}}$ $\quad k_2 = 1\,275\,300{,}0\ \mathrm{N\cdot m^{-1}}$

Damping coefficients: $b_1 = 9\,614{,}0\ \text{kg}\cdot\text{s}^{-1}$

Note: Inherent part of the process of solving the problem is the formulation of simplified models of the vehicle, their mathematical description, and determination of the vehicle's response in the time domain. Computational models of vehicles can have varied complexity depending on the nature of problem to be solved. Oftentimes the so-called quarter- of half models are used; these models model motion and effects of a quarter or half of the vehicle. Nowadays, however, it is not uncommon to use spatial models of vehicles.

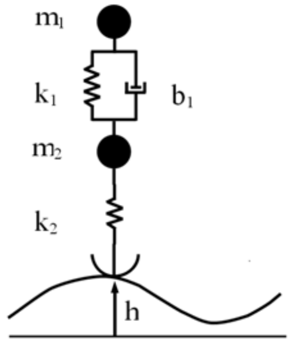

Figure 3: Quarter model of a vehicle

The law of conservation of mechanical energy is a special case of the conservation of energy law, which applies to all types of energy. In the case of dissipative forces such as frictional forces, part of the mechanical energy is converted to heat, but the total amount of energy remains the same.

Solution: Applying a general procedure [2] to the model from Figure 3, we obtain equations of motion of the modeled vehicle. With that we also obtain expressions describing interaction forces at the point of contact of the vehicle's axle with the road surface.

$$r_1''(t)m_1 = \{-k_1[r_1(t)-r_2(t)] - b_1[r_1'(t)-r_2'(t)]\}$$
$$r_2''(t)m_2 = \{+k_1[r_1(t)-r_2(t)] - k_2[r_2(t)-h(t)] + b_1[r_1'(t)-r_2'(t)] - b_2[r_2'(t)-h'(t)]\} \quad (8)$$

Using the principle of equal action and reaction, we derive the following:

$$F(t) = -F_{RV}(t) = -G_2 + k_2[r_2(t)-h(t)] + b_2[r_2'(t)-h'(t)] = F_{st} + F_{dyn}(t),$$

i.e., $F_{st} = -G_2$ and $F_{dyn}(t) = k_2[r_2(t)-h(t)] + b_2[r_2'(t)-h'(t)]$. (9)

We rearrange the equations (8) as follows:

$$m_1 r_1''(t) + b_1 r_1'(t) - b_1 r_2'(t) + k_1 r_1(t) - k_1 r_2(t) = 0$$
$$m_2 r_2''(t) - b_1 r_1'(t) + b_1 r_2'(t) + b_2 r_2'(t) - b_2 h'(t) - k_1 r_1(t) + k_1 r_2(t) + k_2 r_2(t) - k_2 h(t) = 0$$
$$F_{dyn}(t) = b_2 r_2'(t) - b_2 h'(t) + k_2 r_2(t) - k_2 h(t). \quad (10)$$

Function $f(t)$ and its time derivative will be then transformed in this way: $a\,f(t)$ to $a\,F(\omega)$, $f'(t)$ for $f(\pm\infty) = 0$ to $\mathrm{i}\omega F(\omega)$, $f''(t)$ for $f'(\pm\infty) = f(\pm\infty) = 0$ to $-\omega^2 F(\omega)$.

The complex Fourier transform of (Eq. 10) after rearranging has the following form:

$$\bar{r}_1\cdot[-m_1\cdot\omega^2 + \mathrm{i}\cdot b_1\cdot\omega + k_1] + \bar{r}_2\cdot[-\mathrm{i}\cdot b_1\cdot\omega - k_1] = 0$$
$$\bar{r}_1\cdot[-\mathrm{i}\cdot b_1\cdot\omega - k_1] + \bar{r}_2\cdot[-m_2\cdot\omega^2 + \mathrm{i}\cdot b_1\cdot\omega + \mathrm{i}\cdot b_2\cdot\omega + k_1 + k_2] + [-\mathrm{i}\cdot b_2\cdot\omega - k_2] = 0,$$
$$\bar{F}_{dyn} = \bar{r}_2\cdot[\mathrm{i}\cdot b_2\cdot\omega + k_2] + [-\mathrm{i}\cdot b_2\cdot\omega - k_2]. \quad (11)$$

The first two equations of (Eq. 11) can be written as $[a]\cdot\{\bar{r}\} = \{PS\}$ or in the matrix form as

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{Bmatrix} \bar{r}_1 \\ \bar{r}_2 \end{Bmatrix} = \begin{Bmatrix} PS_1 \\ PS_2 \end{Bmatrix} \quad (12)$$

A solution is the found using the Cramer's rule, i.e., $\bar{r}_2 = \dfrac{D_2}{D}$, $\bar{r}_1 = \dfrac{D_1}{D}$, (13)

where $D = a_{11}\cdot a_{22} - a_{12}\cdot a_{21}$, $D_1 = PS_1\cdot a_{22} - PS_2\cdot a_{12}$, $D_2 = a_{11}\cdot PS_2 - PS_1\cdot a_{21}$.

If we then consider that in the Fourier transform the parameter ω represents the angular frequency in $\left[\frac{rad}{s}\right]$, then the coefficients a_{ij} in (Eq. 12) have the following form:
$a_{11} = \left(k_1 - m_1 \cdot \omega^2\right) + \mathrm{i} \cdot \left(b_1 \cdot \omega\right)$, $a_{12} = \left(k_1\right) + \mathrm{i} \cdot \left(-b_1 \cdot \omega\right)$, $PS_1 = 0 + \mathrm{i} \cdot 0$, $a_{21} = \left(k_1\right) + \mathrm{i} \cdot \left(-b_1 \cdot \omega\right)$,
$a_{22} = \left(k_1 + k_2 - m_2 \cdot \omega^2\right) + \mathrm{i} \cdot \left(\left(b_1 + b_2\right) \cdot \omega\right)$, $PS_2 = (k_2) + \mathrm{i} \cdot \left(b_2 \cdot \omega\right)$.

The expression (Eq.13) is calculated numerically for chosen values of ω in the selected frequency band. In this solution we ignore the damping of the tire, i.e. $b_2 = 0$ [kg s^{-1}]. The solution thus applies to the simplified model shown in Fig. 3. Since $b_2 = 0$, we have $\overline{F}_{dyn} = k_2 \cdot \left(\overline{r}_2 - 1\right)$.

5 Conclusion

Why do we use transformations? For various reasons, for instance:

- Transformations allow transforming a complicated problem to a potentially simpler one.
- The problem can be then solved in the transform domain.
- Using the inverse transform we obtain solutions in the original domain.
- Fourier transform is appropriate for periodical signals.
  - It allows uniquely transforming a signal from/to time representation $f(t)$ to/from frequency representation $F(\mathrm{i}\omega)$.
  - It allows analyzing the frequency content (spectrum) of a signal (for instance in non-invasive methods – material diagnostics or magnetic resonance).

The basis of every experimental science is measurement, since it is the only tool to quantitatively describe properties of real-world physical processes. Solution of dynamical problems can be realized both in the time and the frequency domain. Both forms have their advantages, complement one another and represent two different facets of the same physical phenomenon.

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