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SOLVING OF GEOMETRICAL PROBLEMS USING DIFFERENT METHODS

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ABSTRACT. *Geometry is used daily by almost everyone. Geometry is found everywhere: in art, architecture, engineering, robotics, land surveys, astronomy, sculptures, space, nature, sports, machines, cars and much more. Geometry has an important role in the mathematics education process, too. When teaching geometry, spatial reasoning and problem solving skills will be developed. In the early years of geometry the focus tends to be on shapes and solids, then moves to properties and relationships of shapes and solids and as abstract thinking progresses, geometry becomes much more about analysis and reasoning. In this article we present six geometrical problems solved using different methods.*

KEY WORDS: *Geometry. Problem solving. Geometrical constructions.*

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Introduction

Simply written, geometry is the study of the size, shape and position of 2 dimensional shapes and 3 dimensional figures. However, geometry is used daily by almost everyone. Geometry is found everywhere: in art, architecture, engineering, robotics, land surveys, astronomy, sculptures, space, nature, sports, machines, cars and much more.

Geometry takes an important place in the mathematics education process too. When teaching geometry, spatial reasoning and problem solving skills will be developed. In the early years of geometry the focus tends to be on shapes and solids, then moves to properties and relationships of shapes and solids and as abstract thinking progresses, geometry becomes much more about analysis and reasoning [1].

Notes to geometrical constructions

At all school levels in Slovakia the experts in mathematics education recognise several types of mathematical problems which can be specified in one term of the characterization.

These types of problems are *word tasks* – tasks in which relationships among given and studied information are expressed in word formulation; *geometrical constructions* – tasks related to geometrical figures and their constructions in accordance with conditions required to outputs.

To the purpose of this article we put attention to geometrical constructions.

To solve these types of tasks is important to use logical rules which are based on the theorems in such way that we achieve the solution from the given elements. The next specification of solution of the geometrical constructions is in extra steps of process of solutions. The steps are analyse, drawing, proof of construction and discussion. Some of these steps we emphasize in problems below.

It is important to note that there are a few methods how to solve the geometrical constructions:

1. *Method of geometrical locus of points in the plane;*

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2. *Method of geometrical transformations in the plane;*
3. *Algebraic–geometrical method;*
4. *Method of vector geometry and coordinates.*

In school practice when teaching geometry we often choose geometrical problems whose solutions need only one method. However, there are many problems which require different methods in different parts of solution. Then we should talk about the solution of the problems using combined methods. For example we use in some part of the solution the method of geometrical transformation and in other part we solve the tasks using algebraic-geometrical method and after it we use constructional sequences. In this article we show solutions of six geometrical problems in whose solutions we use different of the methods mentioned above.

Selected geometrical problems and their solutions using different methods

Problem 1

On the sides BC, CA and AB of triangle ABC are lying points A', B' and C'. Let points A_1, B_1, C_1 and A_2, B_2, C_2 are images of points A, B, C under the enlargements with the same scale factor k and the centres of enlargements are points C', A', B' and B', C', A'.

Prove that the triangles $A_1B_1C_1$ and $A_2B_2C_2$ have common centroid.

Solution

We draw the triangles pertain to this task (Figure 1). As the scale factor k we choose the number $k = -1$. This choice does not upset the generality of the proof.

For prove we choose the vector method.

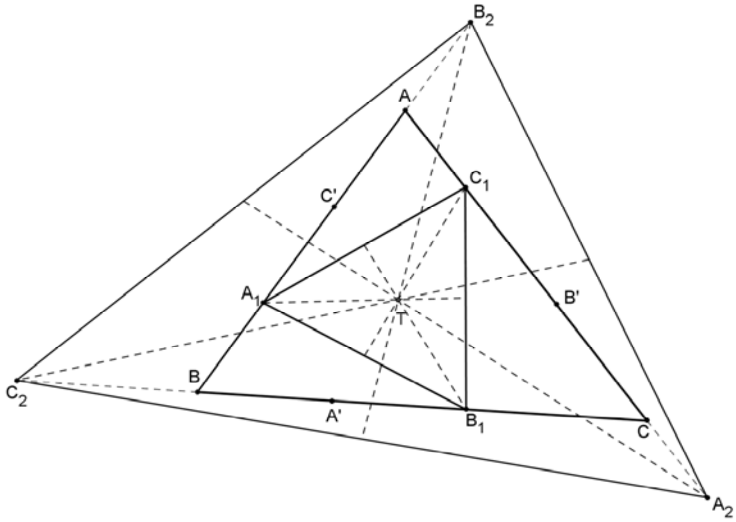

Figure 1

For the points A', B', C', A_1, B_1, C_1 and A_2, B_2, C_2 we can write

$$\overrightarrow{C'A_1} = k \cdot \overrightarrow{C'A}, \quad \overrightarrow{A'B_1} = k \cdot \overrightarrow{A'B}, \quad \overrightarrow{B'C_1} = k \cdot \overrightarrow{B'C}, \qquad \text{(a)}$$
$$\overrightarrow{B'A_2} = k \cdot \overrightarrow{B'A}, \quad \overrightarrow{C'B_2} = k \cdot \overrightarrow{C'B}, \quad \overrightarrow{A'C_2} = k \cdot \overrightarrow{A'C}.$$

Label the centroids of the triangles $A_1B_1C_1$ and $A_2B_2C_2$ as points T_1 and T_2.

Then holds true that:

$$\overrightarrow{T_1A_1} + \overrightarrow{T_1B_1} + \overrightarrow{T_1C_1} = \vec{0}, \qquad (1)$$
$$\overrightarrow{T_2A_2} + \overrightarrow{T_2B_2} + \overrightarrow{T_2C_2} = \vec{0}. \qquad (2)$$

From these implies that

$$\overrightarrow{T_1A_1} = \overrightarrow{T_1C'} + \overrightarrow{C'A_1} = \overrightarrow{T_1C'} + k \cdot \overrightarrow{C'A}$$
$$\overrightarrow{T_1B_1} = \overrightarrow{T_1A'} + \overrightarrow{A'B_1} = \overrightarrow{T_1A'} + k \cdot \overrightarrow{A'B}$$
$$\overrightarrow{T_1C_1} = \overrightarrow{T_1B'} + \overrightarrow{B'C_1} = \overrightarrow{T_1B'} + k \cdot \overrightarrow{B'C}$$
$$\overrightarrow{T_2A_2} = \overrightarrow{T_2B'} + \overrightarrow{B'A_2} = \overrightarrow{T_2B'} + k \cdot \overrightarrow{B'A}$$
$$\overrightarrow{T_2B_2} = \overrightarrow{T_2C'} + \overrightarrow{C'B_2} = \overrightarrow{T_2C'} + k \cdot \overrightarrow{C'B}$$
$$\overrightarrow{T_2C_2} = \overrightarrow{T_2A'} + \overrightarrow{A'C_2} = \overrightarrow{T_2A'} + k \cdot \overrightarrow{A'C}.$$

Using (1) and (2) we obtain

$$\overrightarrow{T_1C'} + \overrightarrow{T_1A'} + \overrightarrow{T_1B'} = k \cdot \left(\overrightarrow{C'A} + \overrightarrow{A'B} + \overrightarrow{B'C}\right) \qquad (1')$$
$$\overrightarrow{T_2B'} + \overrightarrow{T_2C'} + \overrightarrow{T_2A'} = k \cdot \left(\overrightarrow{B'A} + \overrightarrow{C'B} + \overrightarrow{A'C}\right) \qquad (2').$$

For the vectors $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CA}$ holds $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \vec{0}$ and we derive

$$\overrightarrow{AC'} + \overrightarrow{C'B} + \overrightarrow{BA'} + \overrightarrow{A'C} + \overrightarrow{CB'} + \overrightarrow{B'A} = \vec{0}.$$

This means that $\overrightarrow{C'A} + \overrightarrow{A'B} + \overrightarrow{B'C} = \overrightarrow{A'C} + \overrightarrow{B'A} + \overrightarrow{C'B}$.

Now from (1´) and (2´) follows

$$\overrightarrow{T_1C'} + \overrightarrow{T_1A'} + \overrightarrow{T_1B'} = \overrightarrow{T_2B'} + \overrightarrow{T_2C'} + \overrightarrow{T_2A'}$$

That means the points T_1 and T_2 cincide in one point T (see Figure 1).

Problem 2

Divide a segment *AB* into two segments where the ratio of the length of given segment to the length of the longer part is the same as the ratio of the length of the longer part to the length of the shorter part.

Solution

Concider a point C

Let mark the dividing point as $C \in AB$ (as a point on segment *AB*) such that $|AC| > |BC|$. Let $|AB| = a$, $|AC| = x$, then $|BC| = a - x$ (Figure 2).

The problem in algebraical expression is in a form

$$a : x = x : (a - x)$$

Solving

$$x^2 = a \cdot (a - x)$$
$$x^2 + ax - a^2 = 0 \qquad (1)$$

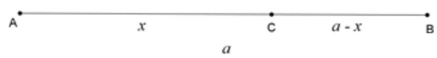

Figure 2

The roots of the equation (1) are

$$x_1 = \frac{1}{2}a\left(\sqrt{5} - 1\right) \qquad (2)$$
$$x_2 = \frac{1}{2}a\left(-\sqrt{5} - 1\right) \qquad (3)$$

The second root is negative that does not satisfy the task.

To construction of the point *C* we modify the expression (2)

$$x_1 = \frac{1}{2}a(\sqrt{5}-1) = \sqrt{a^2 + \left(\frac{1}{2}a\right)^2} - \frac{1}{2}a$$

The segment of length $\sqrt{a^2 + \left(\frac{1}{2}a\right)^2}$ can be constructed using by Pythagorean´ theorem (Figure 3).

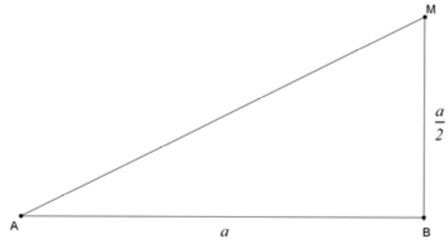

Figure 3

Then we draw segment *x* as we see in Figure 4.

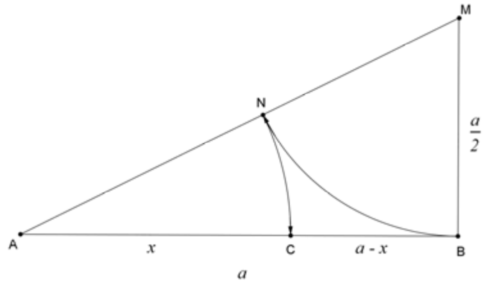

Figure 4

Therefore is valid

$$|BM| = \frac{1}{2}a,$$
$$|MN| = |BM| = \frac{1}{2}a,$$
$$|AM| = \sqrt{a^2 + \left(\frac{1}{2}a\right)^2} = \frac{a}{2}\sqrt{5},$$
$$|AC| = |AN| = |AM| - |MN| = \frac{a}{2}\sqrt{5} - \frac{1}{2}a = \frac{a}{2}(\sqrt{5}-1),$$
$$|BC| = |AB| - |AC| = a - \frac{a}{2}(\sqrt{5}-1) = \frac{a}{2}(3-\sqrt{5}).$$

Now we should express the relevant ratios:

$$\frac{|AB|}{|AC|} = \frac{a}{\frac{a}{2}(\sqrt{5}-1)} = \frac{2}{\sqrt{5}-1} \cdot \frac{\sqrt{5}+1}{\sqrt{5}+1} = \frac{1+\sqrt{5}}{2} = \varphi,$$
$$\frac{|AC|}{|BC|} = \frac{\frac{a}{2}(\sqrt{5}-1)}{\frac{a}{2}(3-\sqrt{5})} = \frac{\sqrt{5}-1}{3-\sqrt{5}} \cdot \frac{3+\sqrt{5}}{3+\sqrt{5}} = \frac{1+\sqrt{5}}{2} = \varphi.$$

We used a combined process during the solution of this task. This combined process we call algebraic – geometric method. This construction originates from Heron.

Note: The ratio φ *is called as golden number. We say that the line segment AB is dividing according to the golden ratio.*

Problem 3

Let be given the line *m*, the length *a*, angle α and a point *V*. Construct a triangle *ABC* in conditions that its side $BC \subset m$, $BC = a$ and $\sphericalangle A = \alpha$.

Solution
Analyse: Let us scatch the triangle *ABC* (we assume that such triangle exists).

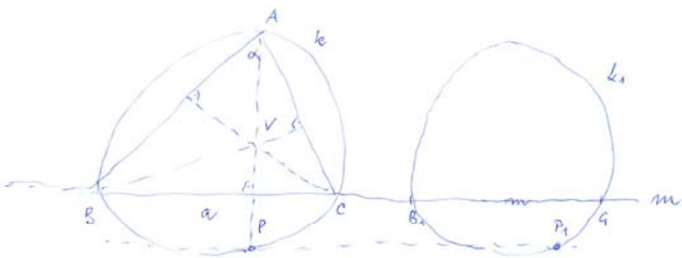

Figure 5

Let k be circumcircle of a the triangle *ABC*. On the circle k there are the points reflected with point V under the reflection of sides of triangle *ABC*. One of these points we label as *P*. This point *P* we could construct from the given objects. The points *A, V, P* are collinear and their line is perpendicular to line *m*. The point *A* is the vertex of the angle (of given value α) at the circumference to the circle *k* pertain to the chord *BC*. The most important is the construction of the circle *k* which helps us to construct all of the vertices of the triangle *ABC*.
Let a translation of the triangle *ABC* which has been translated the side *BC* to B_1C_1 lying on line m. Then the point *P* is translated to P_1 lying on line *m´*. Line *m´* is parallel to line *m* with the circle k has been translated to the circle k_1. The circle k_1 contain the locus of the points at which a line segment *BC* subtends an angle α. Let us concentrate on the construction of the circle k_1.

Construction
1. B_1C_1; $|B_1C_1| = a$
2. k_1; k_1 is the set of all points $X \in p$ such that $|\angle B_1XC_1| = \alpha$
3. P_1; $P_1 \in k_1 \cap m', m' \parallel m, P \in m'$
4. k; $T_{\overrightarrow{P_1P}}: k_1 \to k$
5. $B, C; B, C \in k \cap m$
6. $A; A \in k \cap \overleftrightarrow{VP}$
7. $\Delta\, ABC$

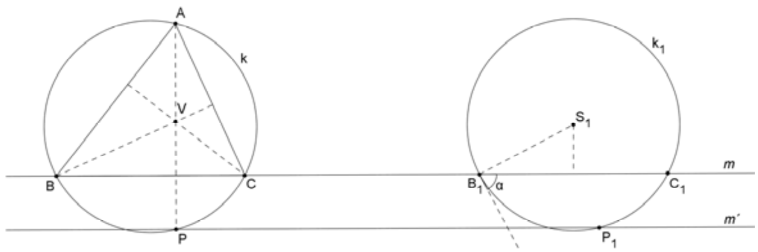

Figure 6

The number of solutions depends of the intersection of the line *m´* and the circle k_1.

Problem 4

Let a triangle ABC. Let points M_1 and M_2 situated on the line BC, $N^{\prime}$ is any point situated on the side CA, $P^{\prime}$ is any point situated on the side AB. Mark the intersection points of the lines $N'M_1$ and $N'M_2$ with side AB as Q_1 and Q_2, the intersection points of the lines $P'M_1$ and $P'M_2$ with side AC as R_1 and R_2. Prove that the lines BC, Q_1R_2, Q_2R_1 cross each other in one point.

Solution

It should be to prove that three lines are concurrent in one point, we prefer to use Menelaus' theorem. The points R_2,Q_1 and I are lying on one line and on the lines containing the sides of the triangle ABC, according to Menelaus´ theorem holds

$$(BCI)\cdot(CAR_2)\cdot(ABQ_1)=1 \qquad (1)$$

Analogically it is true

$$(BCJ)\cdot(CAR_1)\cdot(ABQ_2)=1 \qquad (2)$$

From the collinearity of points we obtain similarly:

$$(BCM_1)\cdot(CAN')\cdot(ABQ_1)=1 \qquad (3)$$
$$(BCM_2)\cdot(CAN')\cdot(ABQ_2)=1 \qquad (4)$$
$$(BCM_1)\cdot(CAR_1)\cdot(ABP')=1 \qquad (5)$$
$$(BCM_2)\cdot(CAR_2)\cdot(ABP')=1 \qquad (6)$$

From (3), (6) and (4), (5) we derive

$$(CAN')\cdot(ABP')\cdot(BCM_1)\cdot(BCM_2)\cdot(CAR_2)\cdot(ABQ_1)=1 \qquad (7)$$
$$(CAN')\cdot(ABP')\cdot(BCM_1)\cdot(BCM_2)\cdot(CAR_1)\cdot(ABQ_2)=1 \qquad (8)$$

From (7) and (8) it follows that

$$(CAR_2)\cdot(ABQ_1)=(CAR_1)\cdot(ABQ_2);$$

From there and (1), (2) it implies that $(BCI)=(BCJ)$. It means that the points I and J merge into one point and therefore the lines BC, Q_1R_2, Q_2R_1 cross each other in one point.

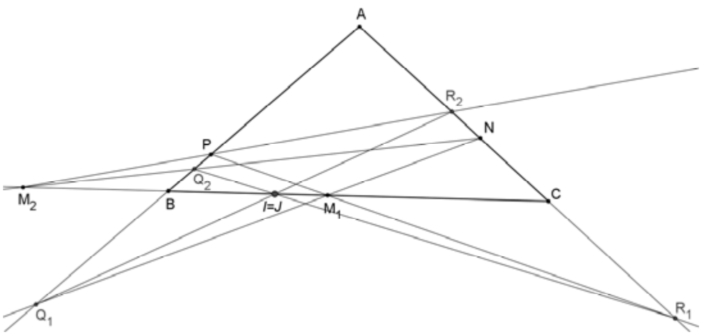

Figure 7

Problem 5

Let line segments of length *a, b, e, f* and an angle of value ε. Construct a quadrilateral $ABCD$ where $|AB|=a$, $|CD|=c$, $|AC|=e$, $|BD|=f$ and $|\angle ASB|=\varepsilon$, while S is the intersection point of the quadrilateral´s diagonals.

Solution

Analyse: We assume that there exists a quadrilateral *ABCD* suitable to the task. Mark the intersection of the diagonals as *S*. In translation $D\rightarrow C$ the point *B* move to point *B´*, the diagonal *BD* move to segment *CB´* parallel to *DB*, then $|\angle ASB'|=\varepsilon$.

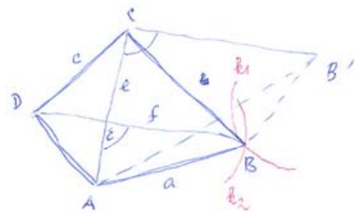

Figure 8

Thus we have concluded: If the quadrilateral has the required properties, then for triangle $AB'C$ we have $|AC| = e$, $|CB'| = f$ and $|\angle ASB'| = \varepsilon$. Based on these information the triangle AB´C is constructible. Then the point B lies on the circles $k_1(B'; c)$ and $k_2(A; a)$. Point D we get in translation $B' \to B$ from point C.

Construction

1. $\Delta\, AB'C$, $|\angle ACB'| = \varepsilon$, $|AC| = e$, $|CB'| = f$
2. k_1; $k_1(B'; c)$
3. k_2; $k_2(A; a)$
4. B; $B \in k_1 \cap k_2$
5. D; $T_{\overrightarrow{B'B}}: C \to D$
6. *ABCD*

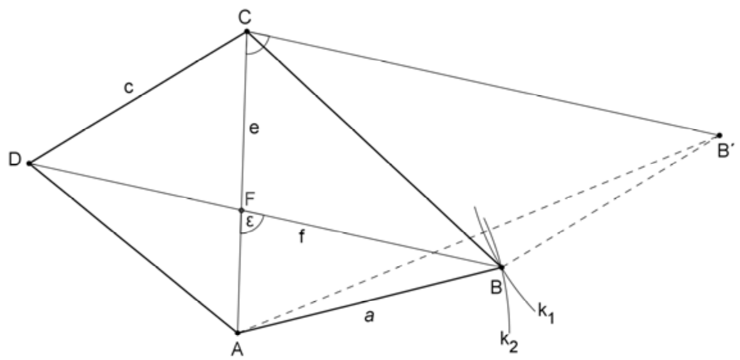

Figure 9

The number of solutions depends of the intersection of the circles k_1, k_2.

Problem 6

Let triangle *ABC*. Construct on its sides points X, Y where $|AX| = |XY| = |YC|$.

Solution

There is not written what kind of triangle is *ABC*. First we solve the task for isosceles triangle *ABC*.

From the symmetry of triangle *ABC* under the perpendicular bisector of the base *AC* follows the parallelism $AC \parallel XY$. The line *AY* intersects the lines *AC*, *XY* under the same alternate angles α. The triangle *AYX* is an isosceles triangle with a base *AY*, the angles connected to the base are congruent. The line *AY* is the bisector of the angle *BAC*.

Construction (Figure 10)
1. $\Delta\, ABC$
2. *m; m* is the bisector of angle *BAC*
3. Y; $Y \in m \cap BC$
4. X; $X \in AB$ and $XY \parallel AC$

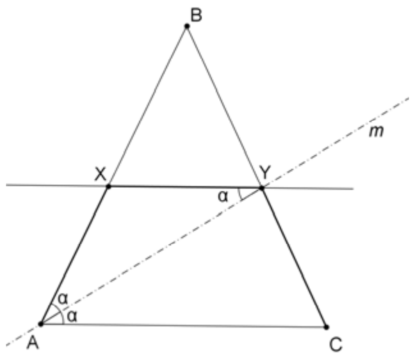

Figure 10

Now we think about a general triangle *ABC*.
We assume that there exist the points *X, Y* suitable to the task.

The points *X, Y* we cannot construct directly, we can construct the points *X′, Y′ C′* in enlargement with the centre *A*. The point *X′* is … lying on *AB* because the scale factor is not done. For the point *Y′* is true: $Y' \in k(X', |AX'|)$ and $Y' \in p$, where p is a parallel to line AC and point $\bar{Y}$ lies on the line p where $\bar{Y} \in BC$ and $|\bar{Y}C| = |AX'|$. The points *X* and *Y* we will found as images of points *X′* and *Y′* under the enlargement with the centre *A* which move the point *C′* to point *C*.

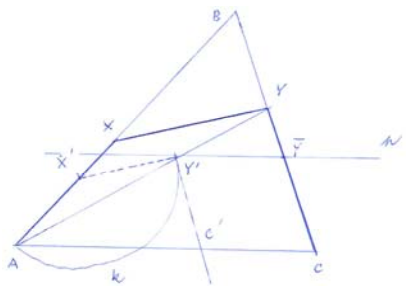
Figure 11

Conclusion

Geometry in general does not belong in students′ opinion to favourite content of mathematics [7]. For students is this area of curriculum very difficult, especially geometrical constructions. That reason calls for more methods to use in solving geometrical problems. Another reason could be as Lester [8] claims „Good problem solvers tend to be more concerned than poor problem solvers about obtaining „elegant“ solutions to problems.

References

[1] Russel, D. *What is Geometry*. Online. Available at http://math.about.com/od/geometry/a/whygeom.htm. Cited on 21 May 2014.
[2] Berger, M. 1984. *Geometria 1, 2.* Nathan, Paris, 1977, 1978 (russian translation: Berže: Geometrija I,II, Moskva Mir 1984) 559 s.
[3] Modenov, P.C. 1980. *Sbornik zadač po specialnomu kurzu elementarnoj matematiky*. Moskva: Vyššaja škola Moskva, 1980.

[4] Kuřina, F. 1996. *10 pohledů na geometrii*. Praha: Matematický ústav Akademie věd České republiky, ISBN 80-85823-21-7

[5] Chmelíková, V. 2009. *Zlatý řez nejen v matematice*. Praha: Katedra didaktiky matematiky MFF UK, 2009. pp. 151—153. ISBN 978-80-7378-078-4

[6] Šedivý, O. 2010. "Ako" a "Prečo" vo vyučovaní matematiky. In: *Konštruktivizmus pri vyučovaní a budovanie geometrických predstáv*. Nitra: UKF v Nitre, 2010. S. 3 -10. ISBN 978-80-8094-723-1

[7] Vidermanová, K. 2007. *Výučba stereometrie a rozvoj priestorovej predstavivosti*, Dissertation theses. Nitra: CPU in Nitra, 2007. 188 p.

[8] Lester, F. K. 1994 Musings about mathematical problem-solving research 1970-1994. In: *Journal for Research in Mathematics Education*, Vol. 25 (1994). P. 660-675. ISSN-0021-8251

[9] Hofstetter, K. 2002. A simple construction of the golden section. In *Forum Geometricorum*, Vol. 2, pp. 65-66.

[10] Yaglom, I. M. 1973. *Geometric transformations III. AMC*, *10*, 12.

[11] Altshiller-Court, N. 1924. College Geometry. *American Mathematical Monthly*, 232-235.

[12] Johnson, R. A. 2007. *Advanced euclidean geometry*. J. W. Young (Ed.). New York: Dover Publications.

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