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NOTES ON SOLUTION OF APOLLONIUS' PROBLEM

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ABSTRACT. *In our paper we concerned with a problem of Apollonius which is signed as CCC in symbolic way in relevant sources. The idea of finding a solution is based on using specific circle inversion in manner which is not conventional. An original construction of the center of inversion causes that inversion maps given circles into three circles of equal radii. This approach substantially simplifies a solution. Some little methodological instructions to using of dynamical software are implemented, too.*

KEY WORDS: *Apollonius' problem, circle inversion, pencils of circles, radical axis*

CLASSIFICATION: *G45, G55, G95*

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Introduction

In our article we discuss one specific solution of the last Apollonius' problem how to draw a circle that is tangent to three given circles in a general position (the circles are not intersecting and none of them is inside the other).

We apply a peculiar method of the circle inversion in a combination with a method of geometrical loci of points. We suppose that the reader is familiar with fundamentals of non-linear mapping such circle inversion and its properties. Further we assume that the reader is acquainted with geometrical concept of radical axis, power of the point with respect to circle, pencils of circles, algebraic equation of circle and finally definition of hyperbola.

The Tangency Problem of Apollonius – short overview

The Problem of Apollonius is one of the most famous geometric problems which was put forth by the greatest scientist of ancient world, Apollonius of Perga (ca. 260 – 170 B.C.) We can formulate it as following:

Consider the problem how to draw a circle subject to three conditions taken from among the following: the circle pass through one or more points, P; to be tangent to one or more lines, L; to be tangent to one or more circles, C. [1].

There are ten problems that can be solved and symbolically are represented by following $PPP, PPL, PLL, LLL, PPC, PLC, PCC, LLC, LCC$ and CCC.

The last one is considered as the hardiest construction problem in general. The *CCC* problem has at most eight possible solutions which drawing can be obtained with different approaches. The methods of finding a solution have been developed by various mathematicians through history, e.g. F. Viete (1540 - 1603), C. F. Gauss (1777 - 1855) or J. D. Gergonne (1771 - 1859). Gergonne's solution is one of the most elegant and is based

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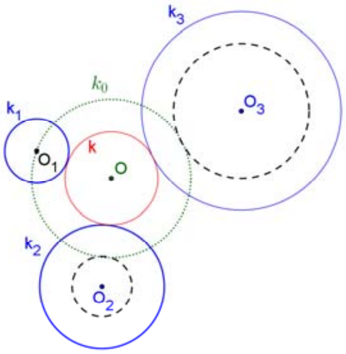
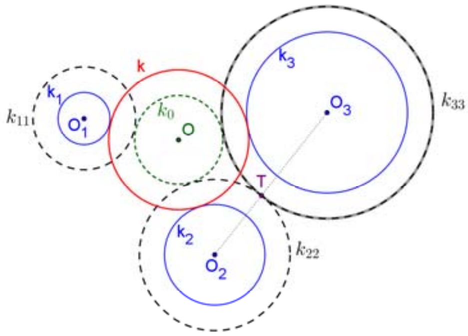
on properties of homothetic circles. The solution is to much extensive for the purpose of this article and so we refer a reader to [2].

Widely used is the method of non-linear mapping called circle inversion. A lot of solutions of the general problem in CCC variant can be reduced to a simpler construction in many ways.

In Fig.1 we suppose that for the given circles $k_1(O_1,r_1)$, $k_2(O_2,r_2)$ and $k_3(O_3,r_3)$ holds $r_1<r_2<r_3$. If we draw concentric circles $k_{22}(O_2,r_2-r_1),k_{33}(O_3,r_3-r_1)$ then the circle $k_0(O,r_0)$ is concentric with a circle k which is a solution. This solution is famous like Viete's reduction of the problem to simpler PCC variant. [3] We note that to find all solutions is necessary to consider with further possibilities of the concentric circles such $k_{22}(O_2,r_2\pm r_1),k_{33}(O_3,r_3\pm r_1)$ [4].



a)



b)

Figure 1: a) Viete's reduction the variant *CCC* to simpler construction problem *PCC,* b) if the center of inversion lies in the point *T* then it leads to a reduction to *PPC* variant.

In [4] one can find a solution in which we draw circles $k_{11}(O_1,r_1+s),k_{22}(O_2,r_2+s),k_{33}(O_3,r_3+s)$, where $s=\dfrac{|O_2O_3|-(r_2+r_3)}{2}$. In this case a circle $k_0(O,r_0)$ is concentric with a circle k. The simplification resides in a moment of the choice of the center of inversion. If it is a tangency point T of the circles k_{22},k_{33} then the inversion maps the circles k_{22},k_{33} onto two parallel lines and the problem is reduced to PPC variant.

In [5] is also described a special construction of circle inversion which maps two non-intersecting circles into a pair of concentric circles. Using this inversion one can map the given three circles $k_1(O_1,r_1)$, $k_2(O_2,r_2)$, $k_3(O_3,r_3)$ into two concentric circles $k_1'(O_1',r_1')$, $k_2'(O_2',r_2')$ and the circle $k_3'(O_3',r_3')$ laying between the circles k_1' and k_2'. To draw a circle k' that is tangent to three circles k_1', k_2', k_3' is a trivial construction task. The circle inversion maps the circle k' into the solution – circle k.

Finally, there are also many other approaches how to solve the problems of Apollonius, including algebraic calculations. There is no place to explain these kinds of solutions in details. We refer the reader to [2, 3, 6, 7, 8, 9].

Theoretical framework

A solution is based on following statement.

Theorem. *The loci of points as centers* S *of circle inversion* $K_i\left[\omega(S,r)\right]$ *which transforms given circles* $k_1(O_1,r_1)$, $k_2(O_2,r_2)$, $O_1 \neq O_2$, $r_1 \neq r_2$ *into circles* $k_1'(O_1',r_0)$, $k_2'(O_2',r_0)$, $O_1' \neq O_2'$ *are two circles* $m_e(O_e,r_e)$ *and* $m_i(O_i,r_i)$. *The points* O_e, O_i *are the centers of homothety of the given circles* $k_1(O_1,r_1)$ *and* $k_2(O_2,r_2)$. [10]

Proof. If an inversion $K_i\left[\omega(S,r)\right]$ maps the circle $k_1(O_1,r_1)$ into a circle $k_1'(O_1',r_0)$ then holds

$$|SO_1'| = \frac{|SP'|+|SQ'|}{2} = \frac{r^2\left(|SP|+|SQ|\right)}{2|SP|.|SQ|} = \frac{r^2}{|SP|.|SQ|}\cdot\frac{|SP|+|SQ|}{2} = \frac{r^2}{|SP|.|SQ|}\cdot|SO_1|,$$

where $P,Q \in O_1O_1' \cap k_1$ and P',Q' their inversion images (Fig. 2)

The product $|SP|\cdot|SQ|$ represents a power h of the point S with respect to the circle $k_1(O_1,r_1)$ [2] and it is easy to derive that

$$\frac{|SO_1'|}{|SO_1|} = \frac{r^2}{|SP|.|SQ|} = \frac{r^2}{(u_1+r_1).(u_1-r_1)}, \text{ where } u_1 = |SO_1|.$$

If the point S lies outside the circle k_1 then the ratio $|SO_1'|:|SO_1|$ is positive and holds

$$\frac{|SO_1'|}{|SO_1|} = \frac{r^2}{|SP|.|SQ|} = \frac{r^2}{(u_1+r_1).(u_1-r_1)} = \frac{r^2}{u_1^2-r_1^2} = \frac{r^2}{h_1}$$

If the point S is an internal point of circle k_1 then $u_1 - r_1 < 0$ and holds

$$\frac{|SO_1'|}{|SO_1|} = \frac{r^2}{|SP|.|SQ|} = \frac{r^2}{(r_1+u_1).(r_1-u_1)} = \frac{r^2}{r_1^2-u_1^2} = -\frac{r^2}{h_1}.$$

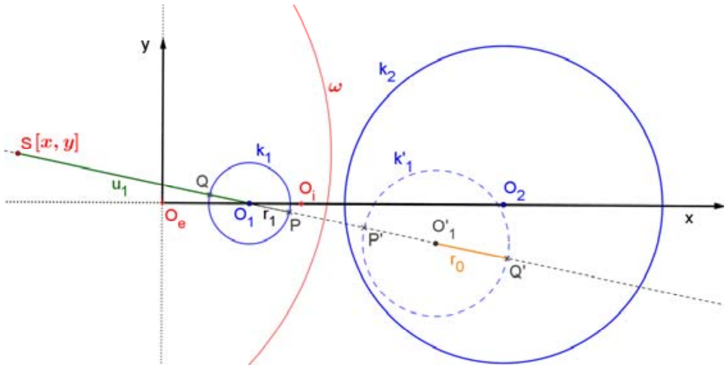


Figure 2 An inversion $K_i\left[\omega(S,r)\right]$ maps the circle $k_1(O_1,r_1)$ into a circle $k_1'(O_1',r_0)$.

[2] The power of a point P with respect to a circle $k(O,r)$ is $h = u^2 - r^2$, $u = |PO|$. By this definition, points inside the circle have a negative power, points outside the circle have positive power and the points on the circle have zero power. [11]

The circles $k_1(O_1,r_1)$, $k_1'(O_1',r_0)$ are also homothetic with the center S and holds true that $\left|SO_1'\right|:\left|SO_1\right| = r_0 : r_1$. We obtain a particular result $\left|SO_1'\right|:\left|SO_1\right| = r_0 : r_1 = r^2 : \left|h_1\right|$.
By analogy we derive $r_0 : r_2 = r^2 : \left|h_2\right|$, where h_2 is the power of point S with respect to the circle $k_2(O_2,r_2)$. From the previous we have $r_1 : r_2 = \pm h_1 : h_2$.

Let us consider O_e as an external center of homothety of the circles $k_1(O_1,r_1)$, $k_2(O_2,r_2)$. From the properties of the homothety $a_1 : a_2 = r_1 : r_2$, where $\left|O_eO_1\right| = a_1$, $\left|O_eO_2\right| = a_2$ and we obtain result $a_1h_2 = \pm a_2h_1$, $h_j = u_j^2 - r_j^2, j = 1,2$.
We put the origin of Cartesian coordinate system into the point O_e in such manner that $O_1[a_1,0]$, $O_2[a_2,0]$, $a_2 > a_1 > 0$.

a) Assume that the point S as an external point of the circles k_1,k_2 or is an internal point of the both of them. Holds true that $a_1h_2 = a_2h_1$. If we label $\left|SO_2\right| = u_2$ then we derive

$$0 = a_1h_2 - a_2h_1 = a_1\cdot\left(u_2^2 - r_2^2\right) - a_2\cdot\left(u_1^2 - r_1^2\right) =$$
$$= a_1\cdot\left[\left(x-a_2\right)^2 + y^2 - r_2^2\right] - a_2\cdot\left[\left(x-a_1\right)^2 + y^2 - r_1^2\right],$$

what can be simplified $$x^2 + y^2 = a_1a_2 - r_1r_2.$$
The last equation represents a circle $m_e(O_e,r_e)$ iff $r_e = \sqrt{a_1a_2 - r_1r_2} > 0$.
If $d = \left|O_1O_2\right|$ then $a_1 = \dfrac{r_1}{r_2 - r_1}d$, $a_2 = \dfrac{r_2}{r_2 - r_1}d$ and holds

$$r_e = \sqrt{a_1a_2 - r_1r_2} = \sqrt{d^2 - \left(r_2 - r_1\right)^2}.$$

This implies that $d > r_2 - r_1$. The condition indicates that the circle k_1 intersects the circle k_2 in two points or the circle k_1 lies outside the circle k_2.

b) Without loss of generality, let us consider the point S as an internal point of the circle k_1 and an external point of the circle k_2.

Holds true that $a_1h_2 = -a_2h_1$ and we calculate

$$0 = a_1h_2 + a_2h_1 = a_1\cdot\left(u_2^2 - r_2^2\right) + a_2\cdot\left(u_1^2 - r_1^2\right) =$$
$$= a_1\cdot\left[\left(x-a_2\right)^2 + y^2 - r_2^2\right] + a_2\cdot\left[\left(x-a_1\right)^2 + y^2 - r_1^2\right].$$

It can be simplified

$$\left(x - 2\frac{a_1a_2}{a_1+a_2}\right)^2 + y^2 = r_1r_2 - a_1a_2 + 4\left(\frac{a_1a_2}{a_1+a_2}\right)^2.$$

This equation represents a circle m_i with a center in the point $O\left[2\dfrac{a_1a_2}{a_1+a_2},0\right]$

providing $$r_i = \sqrt{r_1 r_2 - a_1 a_2 + 4\left(\frac{a_1 a_2}{a_1 + a_2}\right)^2} > 0 .$$

If a point O_i is internal center of the homothety of the circles $k_1(O_1, r_1), k_2(O_2, r_2)$ then for its signed ratio holds $(O_1 O_2 O_i) = \frac{\overrightarrow{O_1 O_i}}{\overrightarrow{O_2 O_i}} = -\frac{a_1}{a_2}$ and we derive $O_i\left[2\frac{a_1 a_2}{a_1 + a_2}, 0\right]$. The center of the circle m_i is the point O_i.

To determine the radius $r_i > 0$ of the circle m_i we obtain similarly

$$r_i = \sqrt{r_1 r_2 - a_1 a_2 + 4\left(\frac{a_1 a_2}{a_1 + a_2}\right)^2} = \ldots = \frac{\sqrt{r_1 r_2}}{r_1 + r_2}\sqrt{(r_1 + r_2)^2 - d^2} .$$

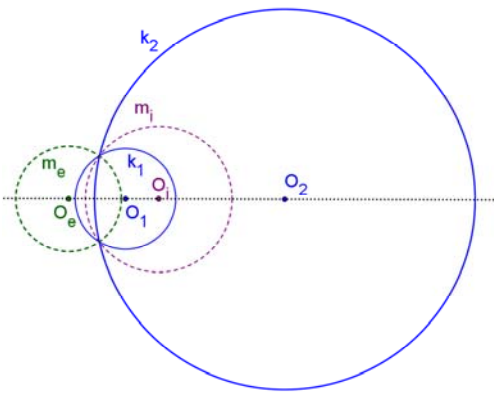
This implies that $0 < d < r_1 + r_2$. The condition means that the circles k_1, k_2 are intersecting or one circle lies inside the other.

Corollary. *The circles* $m_e(O_e, r_e)$, $m_i(O_i, r_i)$ *are circles of a pencil of circles determined with* $k_1(O_1, r_1)$, $k_2(O_2, r_2)$.

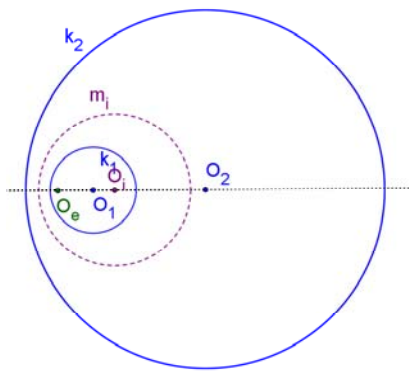
Proof. The statement follows directly from the fact that the equations of the circles $m_e(O_e, r_e)$, $m_i(O_i, r_i)$ can be written by equation $\lambda K_1 + \mu K_2 = 0$, $[\lambda, \mu] \neq [0,0]$, where

$$K_1 = (x - a_1)^2 + y^2 - r_1^2, K_2 = (x - a_2)^2 + y^2 - r_2^2 .$$

If we put $[\lambda, \mu] = [1, -1]$ then we obtain $x = \frac{(a_1 + a_2)}{2a_1 a_2}(a_1 a_2 - r_1 r_2)$. This is an equation of a radical axis of the pencil of the circles $k_1(O_1, r_1)$, $k_2(O_2, r_2)$.



a)



b)

Figure 3 The pencils of the circles in different positions. The circles $m_e(O_e, r_e)$, $m_i(O_i, r_i)$ are the loci of points as centers S of circle inversion $K_i[\omega(S, r)]$ which transforms given circles $k_1(O_1, r_1)$, $k_2(O_2, r_2)$ into congruent circles. If the point $S = m_e \cap k_j, j = 1,2$ (Fig. 3a) then the inversion maps the circles k_1, k_2 into two lines. This special case we will not consider.

Outline of solution of Apollonius' problem in variant *CCC*

Let us find a solution $k_4(O_4,r_4)$ by the method of circle inversion. We will consider a general case, when given three circles k_1,k_2,k_3 have no common points and one lies outside the others. Without loss of generality assume that $r_1<r_2<r_3$, too.

A center S of circle inversion $K_i\left[\omega(S,r)\right]$ we put into an intersection of two circles m_e^{12}, m_e^{23} which are constructed according to the theorem. It holds true that $m_e^{12}\cap m_e^{23}\cap m_e^{31}=\{S_1,S_2\}$.[3] The radius r of the circle ω of the inversion K_i can be chosen arbitrary. If we put $\omega\perp k_3$ then $k_3'\equiv k_3$ and holds true that the inversion $K_i\left[\omega(S,r)\right]$ maps the circles k_1,k_2,k_3 into circles k_1',k_2',k_3 the circles $k_1',k_2',k_3'\equiv k_3$ which have the radius $r_3=r_0$.

It is evident that there exist two circles which are tangent to $k_1'(O_1',r_3),k_2'(O_2',r_3),k_3(O_3,r_3)$. The circles $k_j'(O',r_j'),j=4,5$ have a center O' which is a point of intersection of perpendicular bisectors $O_1'O_2',O_2'O_3$ and $O_1'O_3$ (The point O' is a power point with respect to the circles k_1',k_2',k_3, too.).

The inversion $K_i\left[\omega(S,r)\right]$ maps the circles $k_j'(O',r_j'),j=4,5$ into circles $k_j(O_j,r_j),j=4,5$ which represent two of eight possible solutions.

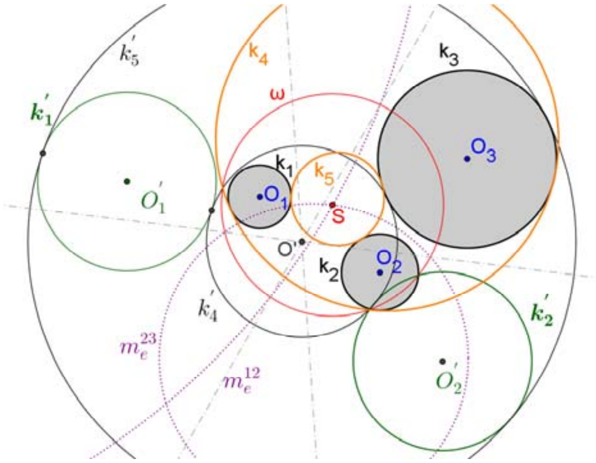


Figure 4 Two of eight possible solutions of the task.

[3] The circles $m_e^{12},m_e^{23},m_e^{31}$ belong to the same coaxal system. It follows from the corollary of the theorem and from the D'Alembert Theorem about the collinearity of centers of homotheties of three given circles. [2, p. 155] Using appropriate dynamical geometry software the reader can illustrate this fact. The rigorous proof is left to the reader as an exercise.

Let us deal with a solution in which a circle $k'(O',r')$ has tangency with the circles k_2', k_3. It is evident that the center O' is a point of the perpendicular bisector $o_{O_2'O_3}$ of the segment $O_2'O_3$ independently on the fact if there exists external or internal tangency.

Let us consider a circle $k'(O',r')$ which has an external tangency with the circle $k_1'\left(O_1',r_3\right)$ and an internal tangency with a given circle $k_2'\left(O_2',r_3\right)$ or vice – versa.

A locus of its centers O' is hyperbola h which focuses are the centers O_1', O_2'. This follows from the definition of a conic section because holds $\left\| O_1'O' \right| - \left| O_2'O' \right\| = 2r_3$. The line $o_{O_2'O_3}$ intersects hyperbola h in a center of the circle k'. There are two solutions there which the circle inversion $K_i\left[\omega(S,r)\right]$ maps into additional two solutions (Fig. 5).

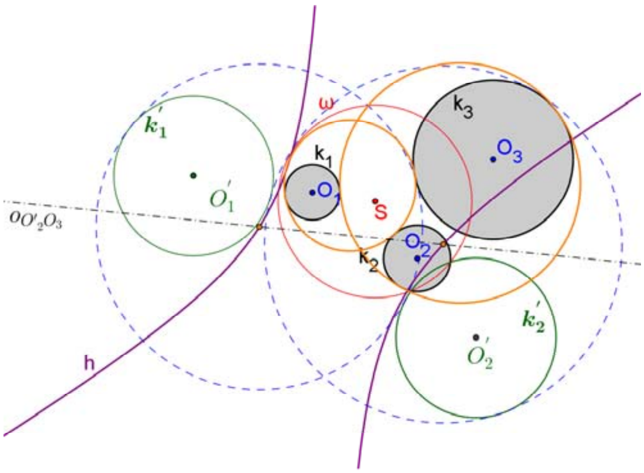


Figure 5 The additional solutions of the Apollonius' problem related to using a hyperbola.

By analogy we can draw the other four solutions if we use the perpendicular bisectors of the segments $O_1'O_3$, $O_1'O_2'$ and an appropriate hyperbola.

Conclusion

Apollonius' problem belongs to one of the most interesting geometrical tasks which solution integrated within theoretical knowledge from different domains of geometry. As we presented in the introduction, widely used is the non-linear transformation called circle inversion. In this article we apply this transformation, too.

The idea of the outlined geometrical solution is in a transition of given circles into congruent circles. As was shown two of eight solutions can be drawn immediately. To complete the solutions in details should enlarge extend of this paper significantly. This is something that can be left to the reader as a practice.

Our last note is related to a practical construction. A drawing all solutions is suitable to use an appropriate dynamic geometry software, mostly due to an interactive construction of conic sections. Some methodological approach can be found in [12, 13].

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