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NOTES TO EXTREMA OF FUNCTIONS

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ABSTRACT. *The necessary and the sufficient condition for the existence of a local extremum give us instructions on how to find extrema and also provide a procedure to determine the kind of extremum. However, the situation may not always be exemplary, as we would expect, and for some tasks (mentioned in the paper) we also have to use other methods in order to detect local or constrained extrema.*

KEY WORDS: *function of several variables, local extrema, constrained extrema*

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Introduction

A traditional part of mathematics courses at universities is mathematical analysis. Its initial parts include also differential calculus. Students learn the concepts of derivative (one-sided, improper), differential, higher order derivatives, etc.
An important continuation of this topic is also the question of application of differential calculus, i.e. mainly tasks dealing with finding local and global extrema of functions. This area has a great use in terms of extrema of functions of one variable and extrema of functions of several variables.
Let us first mention a function of one variable. The existence of an extremum at some point is especially dealt with in Fermat's theorem: *if a function f has its local extremum at a point c and if f is differentiable at c, then* $f'(c)=0$. The character of this extreme is dealt with in the following theorem: *let c be a stationary point of a function f and let* $f''(c)\neq 0$ *; if* $f''(c)>0$ *then f has a sharp local minimum at c, and if* $f''(c)<0$ *then f has a sharp local maximum at c.*
The situation may be more complicated if we consider, for example, the function $f: y=x^{23}$, for which $f'(0)=f''(0)=...=f^{(22)}(0)=0$. However, in the end we find also a non-zero derivative of some order, and depending on its parity we decide about the (non)existence of a local extremum at a suspected stationary point.

Thus we see that even in the case of functions of one variable the situation is not always clear in advance (and we considered only cases where the derivate at the point exists), and that we do not always find a local extremum at a stationary point.

Constrained Local Extrema

Also due to the previous lines it is expected that finding extremes in case of functions of several variables does not have to be only a matter of using an algorithm. Once again, we will at first mention the necessary condition for the existence of extrema: *if a function* $f(x_1,...x_n)$ *has an extremum at a point A, and if all partial derivatives in A exist, then*

$df(A)=\sum_{i=1}^{n}\frac{\partial f(A)}{\partial x_i}dx_i=0$. Let us add a sufficient condition, which is also about the nature of local extremum (for simplicity, let us consider only a function of two variables): *let a point A be a stationary point of a function* $f(x,y)$, *let f be twice differentiable at A, let* $D=f''_{xx}(A)f''_{yy}(A)-\left[f''_{xy}(A)\right]^2$; *if* $D>0$ *and at the same time* $f''_{xx}(A)>0$ *(respectively* $f''_{xx}(A)<0$*), then f has a sharp local minimum (respectively maximum) at A, and if* $D<0$, *f has no local extremum.*

We can immediately guess the first problem – if the determinant $D=0$, we cannot determine the existence of an extremum at the stationary point.
Finding constrained extrema only adds more difficulties. In their analysis we use Lagrangian *L*. If we are looking for constrained extrema of a function $f(x,y)$ subject to the constraint $g(x,y)=0$ we create the following Lagrangian $L(x,y,\lambda)=f(x,y)+\lambda g(x,y)$.
In addition, the following theorem is valid: *if a function L has a local extremum at a point C, then function f has a constrained local extremum at C.* Unfortunately, again, this is only implication and thus we might not find all constrained extrema of *f*.
During Mathematical Analysis seminars students of course prefer standard algorithmic tasks, where one procedure leads to a correct result. We will show several tasks with "difficulties" in the solution. Either the value of the determinant *D* at stationary points is zero or even negative – i.e. Lagrangian will not have a local extremum at the given point even though the original function will have a constrained extremum at the point.

Problem 1.
Find local extrema of the function $u(x,y)=(x-y)^2+(y-1)^3$.
Solution.
Solving the system of equations $u'_x=0$, $u'_y=0$ we find the stationary point $\Xi[1;1]$. However $D(\Xi)=0$ and thus we can't decide whether the function *u* has an extremum at this point.
Let us therefore choose points Λ, Π in the neighborhood of Ξ and examine the difference $\Delta u=u(\Lambda)-u(\Xi)$, respectively $\Delta u=u(\Pi)-u(\Xi)$. Let the points Λ, Π lie on a line $y=x$, then $\Delta u=(y-1)^3$. If we select $y_\Lambda>1$, respectively $y_\Pi<1$, we get $\Delta u=u(\Lambda)-u(\Xi)>0$, respectively $\Delta u=u(\Pi)-u(\Xi)<0$. The expression Δu thus does not maintain the sign, i.e. there is no extremum at the point Ξ.

Problem 2.
Find constrained local extrema of the function $f(x,y)=xy$, if the constriction is given by the condition $x^2+y^2=2$.
Solution.

Let us create the Lagrangian $L(x,y,\lambda)=f(x,y)+\lambda g(x,y)==xy+\lambda\left(x^2+y^2-2\right)$.
Solving the system of equations $L'_x=0$, $L'_y=0$, $L'_\lambda=0$ we get the following stationary points: for the value $\lambda=-\frac{1}{2}$ points $\Gamma[1;1]$, $\Lambda[-1;-1]$; for the value $\lambda=\frac{1}{2}$ points body $\Theta[1;-1]$, $\Omega[-1;1]$.

We can easily verify that unfortunately $D(\Gamma)=D(\Lambda)=D(\Theta)=D(\Omega)=0$. The existence of extrema is therefore verified through examination of the total differential of the second order d^2L.
For the point Γ we get $d^2L(\Gamma)=-dx^2+2dx\,dy-dy^2$. Let us consider points from the neighborhood of Γ which also lie on a tangent to the circle $x^2+y^2=2$ constructed at the point Γ. Then such point $X[dx;dy]$ has to satisfy the condition $g'_x(\Gamma)dx+g'_y(\Gamma)dy=0$, i.e. $2dx+2dy=0$. Let $dy=-dx$. Then $L(X)-L(\Gamma)=dL(\Gamma)+\frac{1}{2}d^2L(\Gamma)=-2\,dx^2<0$, respectively $L(X)<L(\Gamma)$.
Therefore the function L has a local maximum at the point Γ, thus the function f a constrained local maximum.
Let us take the point Θ. It is obvious that $d^2L(\Theta)=dx^2+2dx\,dy+dy^2$. The points $X[dx;dy]$ from the neighborhood of Θ which also lie on a tangent to the circle $x^2+y^2=2$ constructed at the point Θ satisfying the condition $g'_x(\Theta)dx+g'_y(\Theta)dy=0$, i.e. $2dx-2dy=0$. Let $dy=dx$. Then $L(X)-L(\Theta)=dL(\Theta)+\frac{1}{2}d^2L(\Theta)=2\,dx^2>0$, respectively $L(X)<L(\Theta)$.

Therefore the function L has a local minimum at the point Θ, thus the function f a constrained local minimum.

Analogous analysis can show that the function f has at Λ a constrained local maximum and at the same time at Ω a constrained local minimum.

Problem 3.
Find constrained local extrema of the function $u(x,y)=x^2-y^2$, if the constriction is given by the condition $2x-y-3=0$.
Solution.

Let us define the Lagrangian $L(x, y, \lambda) = x^2 - y^2 + \lambda(2x - y - 3)$. Solving the system of equations $L'_x = 0$, $L'_y = 0$, $L'_\lambda = 0$ we get the stationary point $\Phi[2;1]$. However it is true that $D(\Phi) = L''_{xx}(\Phi)L''_{yy}(\Phi) - \left[L''_{xy}(\Phi)\right]^2 < 0$.

Thus the Lagrangian does not have a local extremum at the point. Despite this fact there really is a constrained local extremum of the function *u* at Φ. This can be verified if we express for example the variable y from the constraint and we will look for extrema of a function of one variable $u(x, 2x-3) = x^2 - (2x-3)^2$. Truly, for $x_0 = 2$ it is true that $u'(x_0) = 0$ and at the same time $u''(x_0) < 0$, i.e. the function $u(x, y)$ has constrained maximum at $\Phi[2;1]$.

Problem 4.

Find a cuboid with the largest volume and with a diagonal $u = 2\sqrt{3}$.

Solution.

Obviously, we are trying to find the maximum of the function $V(a, b, c) = abc$, with a constraint $a^2 + b^2 + c^2 = 12$. We are looking for extrema of the Lagrangian $L(a, b, c, \lambda) = abc + \lambda\left(a^2 + b^2 + c^2 - 12\right)$. Solving the system of equations $L'_a = 0$, $L'_b = 0$, $L'_c = 0$, $L'_\lambda = 0$ we get the stationary point $\Upsilon[2;2;2]$, while $\lambda = -1$. However, this time it is true that $D(\Upsilon) = L''_{aa}(\Upsilon)L''_{bb}(\Upsilon) - \left[L''_{ab}(\Upsilon)\right]^2 = 0$.

The character of the extremum will be decided using the second differential, respectively the equality $L(X) - L(\Upsilon) = \frac{1}{2}d^2L(\Upsilon)$. After calculating the second order partial derivatives we obtain $L(X) - L(\Upsilon) = -(dx - dy)^2 > 0$, i.e. the function *f* has a local maximum at Υ. Then the "cuboid" with the largest volume satisfying the condition is in fact a cube with an edge length $a = 2$.

Problem 5.

Find constrained local extrema of the function $w(x, y, z) = xyz$, if the constriction is given by the conditions $x + y - z = 3$, $x - y - z = 8$.

Solution.

Let us create a Lagrangian (this time of five variables) $L(x, y, z, \alpha, \beta) = xyz + \alpha(x + y - z - 3) + \beta(x - y - z - 8)$. Solving the system of equations

$$L'_x = 0, \; L'_y = 0, \; L'_z = 0, \; L'_\alpha = 0, \; L'_\beta = 0$$

we get the stationary point $\Psi\left[\frac{11}{4};-\frac{5}{2};-\frac{11}{4}\right]$. Further calculations however show that $D(\Psi)=L''_{xx}(\Psi)L''_{yy}(\Psi)-\left[L''_{xy}(\Psi)\right]^2<0$, i.e. the Lagrangian does not have an extreme at this point.

But what about the original function *w*? Let us take a look at this problem once again – geometrically. Points lying in the set defined by our two constraints are actually points lying on a line which is the intersection of the two given planes. Its parametric representation is $x=t$, $y=-\frac{5}{2}$, $z=t-\frac{11}{2}$; $t\in\mathbb{R}$.

Then we can look for extrema of the function of one variable $w(t)=xyz=-\frac{5}{2}t^2+\frac{55}{4}t$ .We can easily verify that this function has a local maximum at $t_0=\frac{11}{4}$, i.e. the function $w(x,y,z)$ has a constrained local maximum at Ψ .

Conclusion

In case of both local and constrained extrema we can find – as with other topics – a number of tasks that have a demonstrational “school” character. We use a predetermined algorithm to find stationary points, and then with help of other known rules we determine the type of extremum. This article presents several non-standard tasks where, for various reasons, difficulties with the application of the said algorithm arise. Students are thus forced to meet the main objective of learning mathematics – to think.

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