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PROBLEMS WITH INFINITY

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ABSTRACT. *The various admittances to interpretation of infinity became the source of gravest ideological collisions and philosophical conflicts. Antinomies was adjusted in set theory by the medium of axiomatization. On the other hand there are many mathematical objects that we do not know to imagine. For example, the existence of such several objects arises from simple fact that the sets* $\mathbb{N}, \mathbb{Q}$ *are equivalent. So there is a sequence of all pairwise different rational numbers, like this* $0; -1,1; -2, -\frac{1}{2}, \frac{1}{2}, 2;\ -3, -\frac{1}{3}, \frac{1}{3}, 3; -4, -\frac{3}{2}, -\frac{2}{3}, -\frac{1}{4}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, 4; \cdots (= \{r_n\}_{n=1}^{\infty})$ *defined through the medium the weight of a rational number. It is not possible to discover a common formula, by means of which we would say where exactly a rational number* $\frac{p}{q}$ *is found in the sequence. The function f,* $f(x) = \sum_{n:r_n \le x} 2^{-n}$*, is defined on* $\mathbb{R}$ *correctly, but we do not know to calculate exactly its values. Furthermore, f* is increasing whereby $\mathbb{Q}$ is *the set of its points of discontinuity. It is known about the Lebesgue measure* μ *that* $\mu(\mathbb{Q}) = 0$*. Let* $\{s_n\}_{n=1}^{\infty}$ *be a sequence of all pairwise different rational numbers from the interval* $\langle 0,1 \rangle$ *and let* $J_n = \left(s_n - \frac{1}{5^n}, s_n + \frac{1}{5^n}\right)$*,* $n \in \mathbb{N}$*. The intervals* J_n*,* $n \in \mathbb{N}$*, do not cover* $\langle 0,1 \rangle$ *because* $\mu(\bigcup_{n=1}^{\infty} J_n) \le \frac{1}{2}$*.Furthermore, the set* B*,* $B = \langle 0,1 \rangle - (\bigcup_{n=1}^{\infty} J_n)$*, is uncountable and* $\mu(B) \ge \frac{1}{2}$*. Nevertheless it is hard task to find anywise some element in* B*.*

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Mathematical objects with absent apprehension

We do not commit great imprecision; if we assert that whole contemporary mathematics is in practice based on set theory. This fact was occasioned by ambition to give integrating mathematics some institutional form. By one of most important intention of set theory is the effort on a comprehensive explanation of the notion infinity in mathematics. The various admittances to interpretation of infinity became the source of gravest ideological collisions and philosophical conflicts (see [3]). The fact that we explain sets as arbitrary aggregations of objects guides not only to the exceeding extension and enriching of mathematics but also to some inconveniences. These irritancies consist in overloading various domains of objects examined in traditional mathematical disciplines on new objects that we can´t to measure, but we do not know their also to conceive. We would hesitate at all to accept existence such objects before the admission of set attitude. For example, we refer the curves completely increasing a square or the continuous functions which have not derivative whether the functions which are not possible to define from principled reasons by any formula. It is noteworthy that opinions on similar demands are accustomed to change with time. Many these mathematical objects (that we considered as some pathologic counter-examples or as although entertaining but be not good for anything suitable) can fall into the cynosure on the ground of their theoretical importance otherwise their usefulness in applications. After this manner those fall in the series of "decorous" and "respectable" members of mathematical world. We assign at least one example. The attributes of fractals (geometric objects with non-integer dimension) was regarded per pathological. The fractals are the source of intensive aesthetic experiences thanks to their computerised display today. We find the ideal forms of fractals directly in the country for a wonder at each remove.

Two constructions based on countability of the set of rational numbers

In this paper all constructions make use of the fact, that it is possible to arrange all rational numbers in a sequence. It is possible for example through the medium the weight v of a rational number ($v\left(\frac{p}{q}\right) = |p| + q$, $p \in \mathbb{Z}$, $q \in \mathbb{N}$, p and q are relatively prime):

$$(1) \quad 0; -1, 1; -2, -\tfrac{1}{2}, \tfrac{1}{2}, 2;\ -3, -\tfrac{1}{3}, \tfrac{1}{3}, 3; -4, -\tfrac{3}{2}, -\tfrac{2}{3}, -\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{3}{2}, 4; \cdots .$$

There is the most $2(v-1)$ of rational numbers with a weight v, $v > 1$. It is not possible to discover a common formula, by means of which we would say where exactly a rational number $\frac{p}{q}$ is found in the sequence (1). We say only say that $\frac{p}{q}$ is on some position among first $v(v-1)+1$ $(= 1 + 2 + 4 + \cdots + 2(v-1))$ members, where $v = |p| + q$. It´s well known that every monotone real function defined on the set $\mathbb{R}$ has only a countable set of points of discontinuity. Usually we imagine the set of points of discontinuity M of this function as a set of isolated points. For example, the set M of the function $y = [x]$equals $\mathbb{Z}$, where the symbol $[x]$denotes the whole part of the real number x. The set M can however contain as well infinitely points which are not its isolated points. The function defined on $\mathbb{R}$ by means of formula

$$f(x) = \begin{cases} k, & x = k \in \mathbb{Z} \\ k + \dfrac{1}{n+1}, & x \in \left\langle k + \frac{1}{n+1}, k + \frac{1}{n} \right), k, n \in \mathbb{N} \end{cases}$$

is non-decreasing and $M = \mathbb{Z} \cup \left\{x \in \mathbb{R}; \exists k, n \in \mathbb{N}\ x = k + \frac{1}{n+1}\right\}$. The integers are not isolated points of the set M. We can despite it advance the graph of this function although it consists of infinitely line segments and in addition of infinite set of points. The vision about this function is clear. It is hardly imagine such increasing function defined on $\mathbb{R}$ where $M = \mathbb{Q}$. However such function exists.

Task 1. Let $r_1, r_2, \cdots, r_n, \cdots$ be a sequence of all mutually different rational numbers (for example the sequence (1)). Let us define function f on the set $\mathbb{R}$ by formula

$$f(x) = \sum_{n: r_n \le x} 2^{-n}.$$

We prove accordingly that the function f is increasing and $M = \mathbb{Q}$, where M is the set of all points discontinuity of f..

Solution. It is easy to show that $\sum_{n=1}^{\infty} a_{k_n} < \sum_{n=1}^{\infty} a_n$ for every convergent series $\sum_{n=1}^{\infty} a_n$ with positive members where $\{k_n\}_{n=1}^{\infty}$ is arbitrary increasing sequence of natural numbers which is different from the sequence $\{n\}_{n=1}^{\infty}$. Let us consider series $\sum_{n=1}^{\infty} b_n$, $\sum_{n=1}^{\infty} c_n$, where

$$b_n = \begin{cases} a_n, & n \notin \{k_m; m \in \mathbb{N}\} \\ 0, & n = k_m, \quad m \in \mathbb{N} \end{cases}, \quad c_n = \begin{cases} 0, & n \notin \{k_m; m \in \mathbb{N}\} \\ a_{k_m}, & n = k_m, \quad m \in \mathbb{N} \end{cases}.$$

We have $a_n = b_n + c_n$ for any $n \in \mathbb{N}$ and further

$$\sum_{n=1}^{\infty} a_{k_n} = \sum_{n=1}^{\infty} c_n < \sum_{n=1}^{\infty} b_n + \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty}(b_n + c_n) = \sum_{n=1}^{\infty} a_n.$$

The previous consideration refer also to series $\sum_{n:r_n \le x_1} 2^{-n}$ $(= f(x_1))$, $\sum_{n:r_n \le x_2} 2^{-n}$ $(= f(x_2))$ for every $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$. We conclude that $f(x_1) < f(x_2)$. We have shown that *f* is increasing. We prove yet that the set *M* equals $\mathbb{Q}$. Let $x \in \mathbb{Q}$, $a = r_k$. The following relationships hold for every $x \in \mathbb{R}, x < a$:

$$f(a) = 2^{-k} + \sum_{n:r_n < a} 2^{-n} > 2^{-k} + \sum_{n:r_n \le x} 2^{-n} = 2^{-k} + f(x).$$

By this means $f(x) < f(a) - 2^{-k}$ and

$$\lim_{x \to a^-} f(x) = \sup\{f(x); x < a\} \le f(a) - 2^{-k} < f(a).$$

We have proved that *f* is not continuous in the left at *a*. Furthermore $\lim_{x \to a^+} f(x) = \inf\{f(x); a < x\} \ge f(a)$. We must prove that *f* is continuous in every irrational number *b*. Let us denote s_k the sum first *k* members of series $\sum_{n:r_n < b} 2^{-n}$ $(= f(b))$. So we have $s_k = \sum_{i=1}^{k} 2^{-l_i}$, where $l_1 < l_2 < \cdots < l_k$ is *k* smallest natural numbers with property $r_{l_i} < b$, $i = 1,2,\cdots,k$. Let $r_l = \max\{r_{l_1}, r_{l_2}, \cdots, r_{l_k}\}$. Evidently $r_l < b$. Inequalities $s_k < f(r_l) < f(x) < f(b)$ are true for every $x, r_l < x < b$. So we have that

$$f(b) = \sup\{s_k; k \in \mathbb{N}\} \le \sup\{f(x); x < b\} = \lim_{x \to b^-} f(x) \le f(b)$$

and $\lim_{x \to b^-} f(x) = f(b)$. We prove also that $\lim_{x \to b^+} f(x) = f(b)$. Let $\varepsilon > 0$. Choose such $m \in \mathbb{N}$ that $\sum_{n=m+1}^{\infty} 2^{-n} < \varepsilon$. It is easy to see that there is such real number x_0, $b < x_0$, that $r_i \notin \langle b, x_0 \rangle$, $i = 1,2,\cdots,m$. We conclude that

$$\sum_{n:b < r_n \le x_0} 2^{-n} \le \sum_{n=m+1}^{\infty} 2^{-n} < \varepsilon, f(x_0) = f(b) + \sum_{n:b < r_n \le x_0} 2^{-n} < f(b) + \varepsilon,$$

$$f(b) \le \lim_{x \to b^+} f(x) = \inf\{f(x); b < x\} \le f(x_0) < f(b) + \varepsilon.$$

The equality $\lim_{x \to b^+} f(x) = f(b)$ emerges from arbitrariness positive number ε.

Remark. We do not know exactly to determine the function value $f(x)$ in any point *x*.

It is known about Lebesgue measure μ that $\mu(\mathbb{Q}) = 0$ (see [1]).

Task 2. Let $r_1, r_2, \cdots, r_n, \cdots$ be a sequence of all mutually different rational numbers from the interval $\langle 0,1 \rangle$ (for example, the sequence (1) if we "scratch out" all rational numbers that do not come under $\langle 0,1 \rangle$). Let μ be Lebesgue measure. Let $J_n = \left(r_n - \frac{1}{5^n}, r_n + \frac{1}{5^n}\right)$,

$n \in \mathbb{N}$ and let $B = \langle 0,1 \rangle - (\bigcup_{n=1}^{\infty} J_n)$. Appreciate $\mu(\bigcup_{n=1}^{\infty} J_n)$, $\mu(B)$. Is possible to find some element from the set B?

Solution. Estimate

$$\mu(\bigcup_{n=1}^{\infty} J_n) \leq \sum_{n=1}^{\infty} \mu(J_n) = \sum_{n=1}^{\infty} 2\left(\frac{1}{5}\right)^n = \frac{\frac{2}{5}}{1-\frac{1}{5}} = \frac{1}{2},$$

$$\mu(B) \geq 1 - \mu(\bigcup_{n=1}^{\infty} J_n) \geq \frac{1}{2}.$$

Let *b* be irrational number, $b \in \langle 0,1 \rangle$. We do not know to remove the existence of such r_n that $b \in \left(r_n - \frac{1}{5^n}, r_n + \frac{1}{5^n}\right)$. So we do not know to determine some concrete element in B nevertheless that the set B is uncountable and $\mu(B) \geq \frac{1}{2}$.

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