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ON PENCILS OF LINEAR AND QUADRATIC PLANE CURVES

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ABSTRACT. *There is a range of geometry software tools for teaching and learning many topics in geometry, algebra and calculus from middle school to the university level. Dynamic software package like Geogebra help students to acquire knowledge about many geometric objects both in the classroom and at home. We show basic facts from the theory of pencils of planar curves that can be defined as a linear combination of their equations. In addition, we look at some possibilities of the software GeoGebra to visualize linear and quadratic plane curves, and their linear combinations. In this paper, possible approach how to use the pencil of planar curves in solving classical problems of analytic geometry is discussed.*

KEY WORDS: *Pencil of planar curves, linear combination, dynamic software.*

CLASSIFICATION: *G44, D44*

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1 Introduction

GeoGebra is a multi-platform mathematical software that is simple to use and which is successfully applied at all levels of education in geometry, algebra and calculus. Dynamic geometry software can help students to explore and understand more concepts in geometry on their own. They may see with their own eyes that this or that geometrical theorem is true. By computers with dynamic geometry software we are also able to solve problems which are difficult to solve by classical approach.

In school mathematics, often the applications of a well-known statement are more important than the verification the statement itself. Therefore, we can start with a very simple problem that hopefully both demonstrates the interplay between algebra and geometry and gives motivation to study problems in which understanding of "why it works" is necessary.

Problem 1. Find all solutions to the following system of equations:

$$x+2y-4=0\,;\;\; 3x-5y-1=0 \tag{S}$$

Solution. Multiplying left side of the first equation by 5 and left side of the second by 2, then adding the results and dividing by 11 we obtain $x-2=0$ and, analogously, multiplying left side of the first equation by -3 and adding corresponding sides of both equations and dividing by -11, we have $y-1=0$. Therefore (2, 1) is the solution of (S).

How can we be sure that, both following system of equations

$$x+2y-4=0\,;\;\; x-2=0 \tag{S_1}$$

$$x+2y-4=0\,;\;\; y-1=0 \tag{S_2}$$

are equivalent to (S), i.e. have the same set of solutions as (S)? Using a geometric interpretation of the system (S), the question can be formulated as "why do lines whose equations are $x+2y-4=0$, $x-2=0$ and $y-1=0$ intersect the line $3x-5y-1=0$ at the same point?"

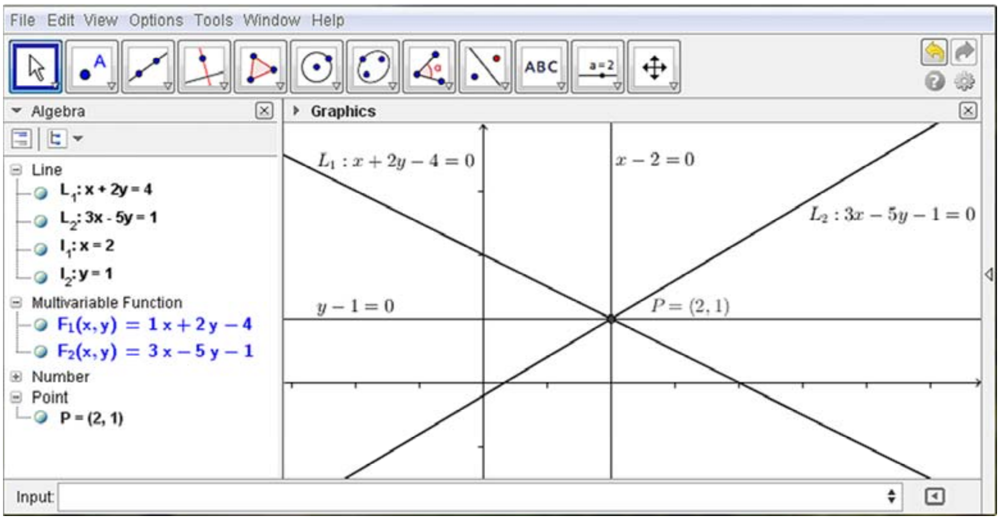

Figure 1

The method we are using above is well-know for everyone. By dynamic software GeoGebra we may verify this method asking a question whether the intersection points of two pairs of lines whose equations are (S_1) and (S_2) are coincident (see Figure 1). Remember that this verification does not replace a proof. We can state the following theorem [1]:

Theorem 1. *Let* $F_1(x,y)$ *and* $F_2(x,y)$ *be two algebraic expressions defined on a set S,* $S\subset R^2$, *and* $\lambda,\mu\in R$, $\mu\neq 0$. *Then the following systems*

$$F_1(x,y)=0, \qquad F_2(x,y)=0\,; \tag{I}$$

$$F_1(x,y)=0, \qquad \lambda F_1(x,y)+\mu F_2(x,y)=0 \tag{II}$$

are equivalent.

In terms of the geometrical interpretation of (I) and (II), the theorem says that the curves defined by the equations $F_2(x,y)=0$ and $\lambda F_1(x,y)+\mu F_2(x,y)=0$ intersect the graph of the curve with the equation $F_1(x,y)=0$ at the same set of points.

When L_1 and L_2 are intersecting lines given by equations $L_1(x,y)=0$, $L_2(x,y)=0$, then the linear combination

$$\lambda L_1(x,y)+\mu L_2(x,y)=0, \tag{1}$$

where λ and μ are parameters not both zero, is the equation of the *pencil of lines* that contain the unique intersection point of L_1 and L_2.

2 Pencil of conics

Let us consider two intersecting lines L_1 and L_2 defined by the equations $L_1(x,y)=0$, $L_2(x,y)=0$. Can we describe the family of all conics passing through the intersection of L_1 and L_2 ? The main idea is to again look at linear combinations of L_1 and L_2, but now, the coefficients will be linear polynomials. That is,

$$\lambda(x,y)L_1(x,y)+\mu(x,y)L_2(x,y)=0 \tag{2}$$

where $\lambda(x,y)$ and $\mu(x,y)$ are polynomials in two variables of degree one, covers all the conics passing through the intersection of L_1 and L_2.

A GeoGebra environment allows the definition of functions of two variables, so we can draw the graph of the implicit function (2). For the polynomials $\lambda(x,y)$ and $\mu(x,y)$, determined by equations of the form $ax+by+c=0$, we can create constants *a*, *b* and *c* with *"Sliders"*. Then we can change the values of *a*, *b* and *c* for both polynomials and see the changed conic described by (2) in both the geometry and algebra windows. In this way, we can determine the family of conics that contains ellipses (including the special case of circles), parabolas, hyperbolas, as well as some degenerate cases such as single point, line, or two lines.

It is also true that the intersection of two distinct conics does not exceed four points. Let C_1 and C_2 be two conics given by equations $C_1(x,y)=0$ and $C_2(x,y)=0$. We call *pencil of conics determined by* C_1 *and* C_2 to the family of conics determined by the quadratic form

$$\lambda C_1(x,y)+\mu C_2(x,y)=0,\text{ where } \lambda,\mu\in R,\ (\lambda,\mu)\neq(0,0). \tag{3}$$

All the conics of the pencil (3) passing through the four intersection points of C_1 and C_2, some of which may coincide or be "imaginary".

The same pencil of conics is gotten by replacing one or both the conics $C_1(x,y)=0$, $C_2(x,y)=0$ by two lines. All these lines are determined by distinct pair of points from the set of intersection of the given conics. If we have exactly four common points of the conics and $L_i(x,y)=0$ be the lines joining pairs of these points, then $L_1(x,y)L_2(x,y)=0$ and $L_3(x,y)L_4(x,y)=0$ are (degenerate) quadrics through the four points. So the equation of the pencil (3) can be written in the form

$$\lambda L_1(x,y)L_2(x,y)+\mu L_3(x,y)L_4(x,y)=0 \tag{4}$$

For any pair $(\lambda,\mu)\neq(0,0)$ of values, we have a conic through the four points determined by the equation pairs $L_i(x,y)=0, L_3(x,y)=0$ and $L_i(x,y)=0, L_4(x,y)=0$, for $i=1,2$.

2 Pencils in problem solving

In this part we will be concerned with the above notion of the pencil in some problems which can be easily solved in analytical geometry using an equation of the pencil of plane curves. First we will solve following problem (see [2]).

Problem 1. Prove that if two given parabolas intersect at four points and their axes are perpendicular, then all common points lie on a circle.

Solution. Let P_1 and P_2 be two parabolas whose axes are perpendicular. We have, in a suitable system of coordinates, the equations of both parabolas P_1 and P_2 in the form:

$$P_1(x,y)=a_1x^2+b_1x+c_1-y=0, \qquad P_2(x,y)=a_2y^2+b_2y+c_2-x=0.$$

With the help of GeoGebra, the configuration of parabolas of given properties has been created (see Figure 2), in which one can use the *"Sliders"* to change the coefficients of polynomials generated both parabolas.

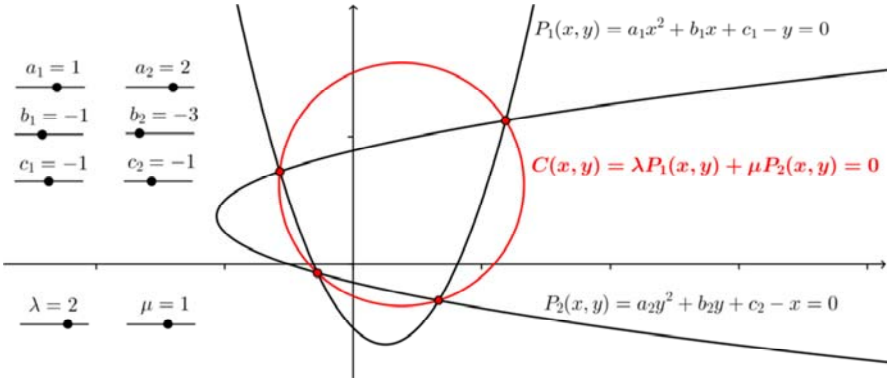

Figure 2

The parabolas P_1 and P_2 give us the pencil

$$\lambda P_1(x,y)+\mu P_2(x,y)=\lambda a_1x^2+\mu a_2y^2+(\lambda b_1-\mu)x+(\mu b_2-\lambda)y+\lambda c_1+\mu c_2=0. \qquad (5)$$

Since $a_1\neq 0$ and $a_2\neq 0$, the equation (5) represents the circle passing through the points of intersection of given parabolas if and only if $\lambda a_1=\mu a_2$. We can set $\lambda=a_2$ and $\mu=a_1$ in (5), and the statement is proven.

To show that the points of intersection lie on a circle is very simple with GeoGebra software. It is sufficient to construct a circle that contains three of the four intersection points. In a next step we use a command *"Relation between Two Objects"* for the last of those points and constructed circle. The program says that the point lies on a circle. The same result we can get by using the *"Sliders"* to change the coefficients λ and μ of the conic (5), as we mentioned above, i.e. $\lambda=a_2$, $\mu=a_1$.

Problem 2. Determine the equation of a conic section passing through three given points $A=(2,0)$, $B=(2,1)$ and $C=(-1,2)$ and whose center is $S=(0,0)$.

Solution. As the point S is the center of the symmetry of the conic we are looking for, we can calculate the symmetric points A' of A and B' of B as follows: $A'=2S-A=(-2,0)$, and $B'=2S-B=(-2,-1)$. We have four points of the conic therefore, we can take the lines $y=0$, $x-2y=0$, $x-2=0$ and $x+2=0$ passing through pairs of the points A, A', B, B'. The pencil of conics has equation

$$\lambda(y)(x-2y)+\mu(x-2)(x+2)=0 .$$

As it contains the point $C=(-1,2)$ we have the following expression:

$$\lambda(2)(-1-2.2)+\mu(-1-2)(-1+2)=0 \quad \Rightarrow \quad 3\mu=-10\lambda$$

If we set $\lambda=3$, than $\mu=-10$, and the equation of the searched conic is

$$10x^2-3xy+6y^2-40=0 .$$

This curve is an *ellipse* with axes not parallel to the coordinate axes (see Figure 3).

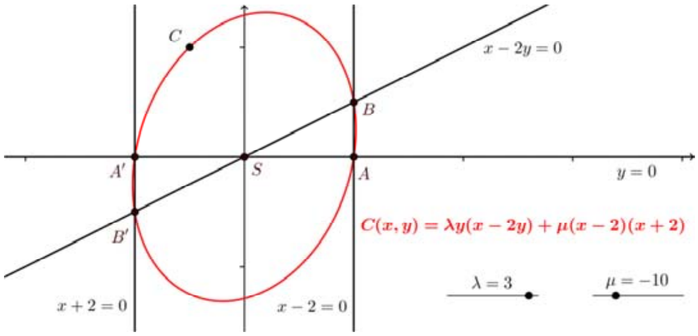

Figure 3

In geometry, we often come across the word *circumcircle* while studying triangles. The circumcircle is the unique circle that passes through each of the triangle's three vertices. We will solve the following problem:

Problem 3. Find the equation of the circumcircle of a triangle whose sides lie on three given lines $L_1(x,y)=x+y-3=0$, $L_2(x,y)=x-y+1=0$ and $L_3(x,y)=x-4=0$.

Solution. Consider a general plane curve given by

$$\lambda L_1(x,y)L_2(x,y)+\mu L_1(x,y)L_3(x,y)+\varepsilon L_2(x,y)L_3(x,y)=0 , \tag{6}$$

where λ, μ and ε are real numbers. Clearly, any such curve passes through the vertices of the triangle. Therefore, the circle we are looking for has equation

$$\lambda(x+y-3)(x-y+1)+\mu(x+y-3)(x-4)+\varepsilon(x-y+1)(x-4)=0$$

for suitable real coefficients λ, μ and ε. This is equivalent to the following equation

$$(\lambda+\mu+\varepsilon)x^2+(\mu-\varepsilon)xy-\lambda y^2-(2\lambda+7\mu+3\varepsilon)x+(4\lambda-4\mu+4\varepsilon)y-3\lambda+12\mu-4\varepsilon=0 .$$

Since $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$ is a circle if and only if $A=C\neq 0$ and $B=0$, and it is not a point or an empty set, λ, μ and ε can be found by equating the coefficients at x^2, y^2 and xy. Hence,

$$\lambda+\mu+\varepsilon=-\lambda\,,\ \ \mu-\varepsilon=0\ \ \Rightarrow\ \ \ \mu=\varepsilon\,,\ \lambda=-\varepsilon\,.$$

If we set $\varepsilon=1$, than $\mu=1$, $\lambda=-1$, and the equation of the searched circle is

$$x^2+y^2-8x-4y+11=0\ \ \Rightarrow\ \ \ (x-4)^2+(y-2)^2=9\,.$$

As shown in Figure 4, all the steps above may be demonstrated in the environment of software GeoGebra.

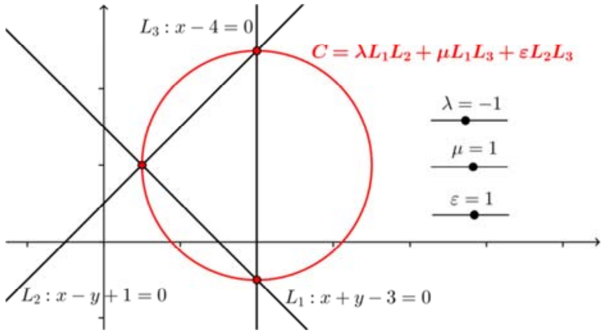

Figure 4

3 Conclusion

The notion of pencil of planar curves that can be defined as a linear combination of their equations is mentioned in this paper. The dynamic software like GeoGebra especially enlightens the connections between algebraic and geometric representation of functions of two variables. In GeoGebra for example it is possible to change the coefficients of linear combination of polynomials determined the pencil of curves. This dynamic point of view allows to students investigate all possible situations for certian family of curves. Although the examples given in this paper are related to the lines and conic sections, the method is also applicable to other plane curves.

References

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