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MORE ON FILTER EXHAUSTIVENESS OF LATTICE GROUP-VALUED MEASURES

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ABSTRACT. *We prove some properties of filter exhaustiveness of lattice group-valued measures and give some characterization in terms of continuity of the limit measure. Furthermore we pose some open problems.*

KEY WORDS: *lattice group, (free) filter, (sequential) filter exhaustiveness, (sequential) continuity of a measure.*

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1 Introduction

The concept of filter exhaustiveness for measures is a powerful and fundamental tool, which has been recently investigated to prove several versions of limit theorems for lattice and topological group-valued measures with respect to filter convergence and some equivalence results (see for instance [3, 4, 5, 7], [8] and the bibliography therein), some comparison results with different kinds of filter convergence (see also [1]), and some results on weak filter compactness and weak filter convergence of measures (see also [2]). In general, it is impossible to obtain results of this kind analogous to the classical ones in the setting of filter convergence, but under the condition of filter exhaustiveness it is possible to give some results about uniform (s)-boundedness, countable additivity or regularity of subsequences, whose elements are indexed by suitable sets of the filter involved.

In this paper we continue the investigation on these topics in the setting of lattice group-valued measures, giving some properties of weak filter exhaustiveness for measure sequences and comparison results, relating them with modes of filter (sequential) continuity investigated in [6] and absolute continuity (see also [10]). In particular, we extend to the context of weak filter exhaustiveness of lattice group-valued measures earlier results of [11], and give some necessary and sufficient conditions for continuity of the limit measure. Moreover we investigate some relation with the continuity of the limit measure of a suitable filter pointwise convergent sequence of measures, and give some equivalence results. Finally, we pose some open problems.

2 Preliminaries

Let G be any abstract nonempty set, R be a Dedekind complete (ℓ)-group, $\mathcal{F}$ be a free filter of $\mathbb{N}$, $\Sigma \subset \mathcal{P}(G)$ be a σ-algebra, and $\nu : \Sigma \to [0,+\infty)$, $m_n : \Sigma \to R$, $n \in \mathbb{N}$, be finitely additive measures. Let d_ν be the pseudometric associated with ν, namely $d_\nu(A,B) = |\nu(A) - \nu(B)|$ for each A, $B \in \Sigma$.

Definitions 2.1 (a) An (O)-*sequence* is a decreasing sequence $(\sigma_p)_p$ in R, whose infimum is 0.

(b) A sequence $(x_n)_n$ in R $(O\mathcal{F})$-*converges* to $x \in R$ iff there is an (O)-sequence $(\sigma_p)_p$ in R with $\{n \in \mathbb{N} : |x_n - x| \le \sigma_p\} \in \mathcal{F}$ for every $p \in \mathbb{N}$.

(c) A sequence $(m_n)_n$ is said to be $(RO\mathcal{F})$-*convergent to* m iff there exists an (O)-sequence $(\sigma_p)_p$ with $\{n \in \mathbb{N} : |m_n(E) - m(E)| \le \sigma_p\} \in \mathcal{F}$ for every $p \in \mathbb{N}$ and $E \in \Sigma$ (see also [9]).

We now deal with filter exhaustiveness for lattice group-valued measure sequences and continuity for (ℓ)-group-valued measures. Further extensions and developments of these concepts will be given in a forthcoming paper.

Definition 2.2 Let $E \in \Sigma$ be fixed. We say that $(m_n)_n$ is ν-*weakly* $\mathcal{F}$-*exhaustive at* E iff there is an (O)-sequence $(\sigma_p)_p$ such that for any $p \in \mathbb{N}$ there is a positive real number δ such that for any $A \in \Sigma$ with $d_\nu(E, A) \le \delta$ there is a set $V \in \mathcal{F}$ with $|m_n(E) - m_n(A)| \le \sigma_p$ whenever $n \in V$.

Definition 2.3 We say that $(m_n)_n$ is *sequentially* ν-*weakly* $\mathcal{F}$-*exhaustive at* E iff there exists an (O)-sequence $(\sigma_p)_p$ such that for each sequence $(E_k)_k$ in Σ with $\lim_k \; d_\nu(E_k, E) = 0$ and $p \in \mathbb{N}$ there are a sequence $(C_k)_k$ in $\mathcal{F}$ and a set $D \in \mathcal{F}$ with $|m_n(E_k) - m_n(E)| \le \sigma_p$ whenever $k \in D$ and $n \in C_k$.

Definitions 2.4 (a) A finitely additive measure $m : \Sigma \to R$ is ν-*continuous* at E iff there is an (O)-sequence $(\sigma_p)_p$ in R such that for any $p \in \mathbb{N}$ there is $\delta > 0$ with $|m(A) - m(E)| \le \sigma_p$ whenever $A \in \Sigma$ and $d_\nu(A, E) \le \delta$.

(b) We say that m is ν-*absolutely continuous on* Σ iff there is an (O)-sequence $(\sigma_p)_p$ such that for every $p \in \mathbb{N}$ there is a positive real number δ with $|m(A)| \le \sigma_p$ whenever $\nu(A) \le \delta$.

Definition 2.5 A finitely additive measure $m : \Sigma \to R$ is said to be *sequentially* ν-$\mathcal{F}$-*continuous at* E iff there is an (O)-sequence $(\sigma_p)_p$ such that for every sequence $(E_k)_k$ in Σ with $\lim_k \; d_\nu(E_k, E) = 0$ we have $(O\mathcal{F})$ $\lim_k m(E_k) = m(E)$ with respect to $(\sigma_p)_p$.

Remark 2.6 Recall that, given a fixed free filter F of N and an element $E \in \Sigma$, a finitely additive measure $m : \Sigma \to R$ is ν-continuous at E if and only if it is sequentially ν-F-continuous at E (see also [6, Theorem 2.1]).

3 The main results

We begin with the following result, which extends [11, Proposition VII.9] to filter exhaustiveness and lattice groups.

Proposition 3.1 *Let $\nu : \Sigma \to [0,+\infty)$ and $m_n : \Sigma \to R$, $n \in \mathsf{N}$, be finitely additive measures, and $E \in \Sigma$. Then the following are equivalent:*

(a) $(m_n)_n$ is ν-weakly F-exhaustive at E;

(b) $(m_n)_n$ is ν-weakly F-exhaustive at $\emptyset$.

Proof: (a) $\Rightarrow$ (b) Let $E \in \Sigma$, $(\sigma_p)_p$ be an (O)-sequence and $\delta > 0$ be according to ν-weak F-exhaustiveness of $(m_n)_n$ at E. Choose arbitrarily $A \in \Sigma$, with $\nu(A) \le \delta$. Since ν is finitely additive and positive, then ν is monotone too, and hence we get:

$$\nu((E \cup A)\Delta E) = \nu(A \setminus E) \le \nu(A),$$
$$\nu((E \setminus A)\Delta E) = \nu(E \setminus (E \setminus A)) = \nu(E \cap A) \le \nu(A).$$

Let $V \in \mathsf{F}$ be in correspondence with E and ν-weak F-exhaustiveness. Taking into account finite additivity of the m_n's, from (1) for every $n \in V$ we have

$$| m_n(A) | = | m_n(E \cup A) - m_n(E \setminus A) | \le$$
$$| m_n(E \cup A) - m_n(E) | + | m_n(E) - m_n(E \setminus A) | \le 2\sigma_p.$$

(b) $\Rightarrow$ (a) Let $(\tau_p)_p$ be an (O)-sequence and δ be according to ν-weak F-exhaustiveness of $(m_n)_n$ at $\emptyset$. Pick arbitrarily $E \in \Sigma$, and let $D \in \Sigma$ be with $\nu(E\Delta D) \le \delta$. Then,

$$\nu(E \setminus D) + \nu(D \setminus E) = \nu(E\Delta D) \le \delta,$$

and a fortiori $\nu(E \setminus D) \le \delta$ and $\nu(D \setminus E) \le \delta$. Let V be associated with $E \setminus D$ and $D \setminus E$, thanks to ν-weak F-exhaustiveness of $(m_n)_n$ at $\emptyset$. For every $n \in V$ we have

$$|m_n(E)-m_n(D)|=|m_n(E\setminus D)+m_n(D\setminus E)|\le$$
$$|m_n(E\setminus D)|+|m_n(D\setminus E)|\le 2\tau_p,$$

getting the assertion.

Analogously as Proposition 3.1, it is possible to prove the following

Proposition 3.2 *Under the same notations and hypotheses as above, the following are equivalent:*

(a) $(m_n)_n$ is sequentially ν-weakly F-exhaustive at E;

(b) $(m_n)_n$ is sequentially ν-weakly F-exhaustive at $\emptyset$.

Remark 3.3 Observe that, arguing analogously as in Proposition 3.1, it is possible to see that a finitely additive measure $m:\Sigma\to R$ is ν-continuous at some set $E\in\Sigma$ if and only if m is ν-continuous at $\emptyset$ if and only if m is ν-absolutely continuous on Σ, and that in this case it is possible to find a single (O)-sequence $(w_p)_p$, independent of $E\in\Sigma$, with respect to which m is ν-continuous at every $E\in\Sigma$.

We now turn to the following characterization of filter exhaustiveness in terms of continuity of the limit measure.

Theorem 3.4 *Let* $E\in\Sigma$, $\nu:\Sigma\to[0,+\infty)$, $m_n:\Sigma\to R$, $n\in\mathbb{N}$, *be finitely additive measures,* $(RO\mathsf{F})$*-convergent to* $m:\Sigma\to R$ *with respect to an* (O)*-sequence* $(\sigma^*_p)_p$*, and fix* $E\in\Sigma$.

Then the following are equivalent:

(a) $(m_n)_n$ is sequentially ν-weakly F-exhaustive at E;

(b) $(m_n)_n$ is ν-weakly F-exhaustive at E.

(c) m is ν-continuous at E.

Proof: (a) $\Rightarrow$ (c) Let $(\sigma_p)_p$ be an (O)-sequence associated with sequential ν-weak F-exhaustiveness of $(m_n)_n$ at E and $(RO\mathsf{F})$-convergence of $(m_n)_n$ to m respectively, and $(E_k)_k$ be a sequence in Σ, with $\lim_k d_\nu(E_k,E)=0$. In order to prove ν-continuity of m at E, thanks to Remark 2.6 it is enough to show that the sequence $(m(E_k))_k$ $(O\mathsf{F})$-converges to $m(E)$ with respect to the (O)-sequence $(2\sigma^*_p+\sigma_p)_p$.

Choose arbitrarily $p \in \mathbb{N}$. Let $D \in \mathcal{F}$ and $(C_k)_k$ be in $\mathcal{F}$ be according to sequential ν-weak $\mathcal{F}$-exhaustiveness, and pick arbitrarily $k \in D$. By $(RO\mathcal{F})$-convergence of $(m_n)_n$ there is a sequence $(B_k)_k$ of elements of $\mathcal{F}$ with

$$| m_n(E) - m(E) | \vee | m_n(E_k) - m(E_k) | \le \sigma_p^* \quad \text{for any } n \in B_k .$$

For each $n \in B_k \cap C_k$ we have

$$| m(E_k) - m(E) | \le | m_n(E_k) - m(E_k) | + | m_n(E_k) - m_n(E) | + | m_n(E) - m(E) | \le 2\sigma_p^* + \sigma_p ,$$

getting the assertion.

(c) $\Rightarrow$ (a) By Remark 2.6, m is sequentially ν-$\mathcal{F}$-continuous at E, and hence there exists an (O)-sequence $(\tau_p)_p$ in R such that for every $p \in \mathbb{N}$ and for each sequence $(E_k)_k$ in Σ with $\lim_k \ d_\nu(E_k, E) = 0$ there is $D \in \mathcal{F}$ with $| m(E_k) - m(E) | \le \tau_p$ whenever $k \in D$. By $(RO\mathcal{F})$-convergence of $(m_n)_n$ to m with respect to the (O)-sequence $(\sigma_p^*)_p$, in correspondence with p, E_k, $k \in \mathbb{N}$, and E there exists a sequence $(F_k^*)_k$ in $\mathcal{F}$ with

$$| m_n(E_k) - m(E_k) | \vee | m_n(E) - m(E) | \le \sigma_p^*$$

for any $n \in F_k^*$. Thus for every $k \in D$ and $n \in C_k$ we get

$$| m_n(E_k) - m_n(E) | \le | m_n(E) - m(E) | + | m(E_k) - m(E) | + | m_n(E_k) - m(E_k) | \le 2\sigma_p^* + \sigma_p ,$$

namely sequential ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at E.

(b) $\Rightarrow$ (c) Let $(\sigma_p)_p$ be an (O)-sequence associated with ν-weak $\mathcal{F}$-exhaustiveness at E, and pick arbitrarily $p \in \mathbb{N}$. By hypothesis there exists a positive real number δ, fulfilling the condition of ν-weak $\mathcal{F}$-exhaustiveness. Fix arbitrarily $A \in \Sigma$ with $d_\nu(A, E) \le \delta$: there exists a set $F_1 \in \mathcal{F}$ with $| m_n(A) - m_n(E) | \le \sigma_p$ for each $n \in F_1$. Moreover, there exists $F_2 \in \mathcal{F}$ with

$$| m_n(A) - m(A) | \vee | m_n(E) - m(E) | \le \sigma_p^*$$

for any $n \in F_2$. For each $n \in F_1 \cap F_2$ we get

$$|m(A)-m(E)| \le |m_n(A)-m(A)|+|m_n(A)-m_n(E)|+|m_n(E)-m(E)| \le 2\sigma_p^* + \sigma_p .$$

Hence $|m(A)-m(E)| \le 2\sigma_p^* + \sigma_p$ for each $A \in \Sigma$ with $d_\nu(A,E) \le \delta$, getting ν-continuity of m at E.

(c) $\Rightarrow$ (b) By ν-continuity of m at E there exists an (O)-sequence (τ_p) in R such that for every $p \in \mathbb{N}$ there is $\delta > 0$ with $|m(A)-m(E)| \le \tau_p$ for every $A \in \Sigma$ with $d_\nu(A,E) \le \delta$. By $(RO\mathcal{F})$-convergence of $(m_n)_n$ to m with respect to the (O)-sequence $(\sigma_p^*)_p$, in correspondence with p, A and E there exists a set $F^* \in \mathcal{F}$ with $|m_n(A)-m(A)$

$$|m_n(A)-m_n(E)| \le |m_n(A)-m(A)|+|m(A)-m(E)|+|m_n(E)-m(E)| \le 2\sigma_p^* + \tau_p ,$$

namely ν-weak $\mathcal{F}$-exhaustiveness of $(m_n)_n$ at E.

Open problems:
(a) Investigate similar notions of filter exhaustiveness and filter continuity, and prove some related comparison results.
(b) Find different results for measures or functions taking values in different types of abstract structures (for example, metric semigroups or topological groups).
(c) Find similar results considering weaker kinds of convergence.
(d) Study similar results requiring R to be super Dedekind complete or even an arbitrary (ℓ)-group.

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