

SELECTED GEOMETRICAL CONSTRUCTIONS

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ABSTRACT. This article deals with selected classical geometric constructions that should be interesting also at the time of modern information and communication technologies. Golden ratio, harmonic proportions, arithmetic and geometrical proportions were already known ratios in the past. Their constructions have been well-known for several centuries. These and many other constructions should not be removed from school curriculums because of their close connection to everyday life. The end of the article is dedicated to a construction, which was in the often used as a layout for building houses last millennium. It is a pity that nowadays these easy constructions have nearly disappeared from common practice. We believe that just these constructions could at this time show closer interconnection between geometry and algebra to students, and thus outline the correlation between synthetic and analytical geometry.

KEY WORDS: geometric construction, golden ratio, Pythagorean theorem, harmonic proportion, arithmetic and geometric proportions

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Introduction

In this age of ICT technologies and the option of graphic software, such as Cabri geometry or GeoGebra, we must not forget that the classical approach to teaching mathematics and geometry should not be completely suppressed by these new technologies.

In doing so, we believe that just these classical constructions could show the students closer interconnection between geometry and algebra and thus could show the students the correlation between synthetic and analytical approach of teaching geometry. Both of these approaches are very important for teaching geometry. Let us demonstrate this on the following theorem:

"Every two conics of the quadratic surface always lie in two conical surfaces."

This theorem was proved using the synthetic method for complexification of a projective space over the field of real numbers. In article [5], we used analytical methods to show that every two conics on the quadratic surface do not always lie on two conical surfaces. The analytical approach allowed us to improve the synthetic one. We discovered a case, when two mutually tangent regular conics, lie on one "simple" conical surface that falls within a sheaf of the quadratic surfaces designated by such conics.

It is apparent that just analytical approach allows us to examine the properties of objects, which we never find by the synthetic approach. We think it is necessary to constantly develop both of these approaches during the whole educational process. Although in the lower grades synthetic approach prevails, the aim should be that students in higher grades know how to solve a complex geometrical problem, utilizing the entire system, i.e. these approaches are not separated, but closely interconnected. Probably every geometry and mathematics teacher wishes that students cope with a geometric situation, so that they are able to plot it and equally well describe it analytically, plus that they know

how to determine which approach offers the easiest solution. In our opinion, the following construction could be helpful in doing so.

Golden ratio

Even in the last millennium, people considered basic geometrical constructions necessary and useful, as an evidenced serves the statement by a German mathematician, physicist and astronomer Johannes Kepler (1571-1630):

"There are two treasures of geometry: Pythagoras' theorem and golden ratio. The first has the prize of gold, the other resembles a precious stone."

We have to realize that the ancient knowledge was the basic foundation of modern knowledge and science. Without wisdom and knowledge of classical geometry, there would be no advances of our time, in the form of graphic software. It is true that these provide us completely different options in the teaching of geometry, for example, give us the opportunity to interactively show and demonstrate certain terms. It is, in our opinion, a great asset, but we would consider it a shame if classical methods of teaching geometry completely disappeared.

From this perspective, some geometrical constructions were forgotten over time, especially concerning mathematical studies at secondary schools and universities. Students rarely learn about the golden ratio of a line segment and other important proportions, or about line segments of irrational and rational length. Students should be encountered with plotting these line segments even when plotting on a number axis in a plane or in space ([4]).

The first clear definition of number φ , which was later called the golden ratio, was brought by Euclid around 300 B.C. He formulated a task, which in today's mathematical language could be read as follows: Divide the line segment AB into two segments - longer AC and shorter CB, so that the area of a square with a side AC is equal the area of a rectangle with sides AB and CB.

This task can be transformed into another task: Divide the line segment AB, so that the ratio of the length the longer part to the shorter is equal the ratio of the entire length of the line segment to the longer part.



Figure 1: Line segment AB divide by golden ratio

If we use the label as in Fig. 1 the task can be written in the mathematically language of algebra in the form of |AC|:|CB|=|AB|:|AC|. Without any loss of generality, we can assume that the length of line segment *CB* is |CB|=1 and the length of line segment *AC* is *x* units, |AC|=x. Thus can be concluded that x:1=(x+1):x. If we find the length of *x*, the task will be solved. After elementary adjustments, we get that sought length *x* is the solution of the quadratic equation $x^2 - x - 1=0$.

Solutions of this equation are numbers $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$.

We sought only positive solution to this equation. The searched length of x is given explicitly, i.e. $x = x_1 = \varphi$, which is the value of the golden ratio and we may conclude that the point C divides the line segment AB by golden ratio [2]. Geometrically, we can

construct the point C as shown in Fig. 2. However, this is not the only possible construction of the golden ratio.



Figure 2: Construction of golden ratio

Harmonic, arithmetic and geometric proportions

Other important ratios that can be considered are arithmetic, geometric and harmonic proportions.

It is a division of a line segment into two parts, so that the length b of the longer part of this line segment is the arithmetic, geometric, or harmonic proportion to the length a of the entire line segment and c the remaining part of this line segment.

The ratios mentioned are now referred to as Pythagorean ratios. If we denote three members *a*, *b* and *c*, where a > b > c, then the member *b* is obtained as arithmetic $b = \frac{a+c}{2}$, $b = \sqrt{ac}$ geometric or harmonic $b = \frac{2ac}{a+c}$ average of the members *a* and *c*, i.e. members *a*, *b*, *c* are three consecutive members of an arithmetic, geometric or harmonic sequence.

If we divide a line segment using the arithmetic proportion, the ratio between the longer part and the shorter part is 2:1, the geometric proportion divides the line segment in the Golden Ratio which we have already mentioned, i.e. the ratio between the longer and the shorter part of this segment is $\frac{\sqrt{5}+1}{2}$: 1 (φ : 1). By using harmonic proportion the ratio between the length of the longer and the shorter part is $\sqrt{2}$: 1. Construction of the dividing point of the line segment through of all the ratios is shown in Fig. 3, point *H* denotes the dividing point of line segment in harmonic proportion, point *G* in a geometric proportion (Golden Ratio) and point *A* in arithmetic proportion (see article [3]).



Figure 3: Division of a line segment in arithmetic, geometric and harmonic proportions (from [3])

In the past, these ratios determined the directions of not only music, but also architecture and it is interesting that nowadays students are not able to do these basically simple constructions. No wonder, because nowadays students do not know how to do the constructions of $\sqrt{2}$, $\sqrt{3}$ by using so-called Pythagorean triangles or by using Euclid's theorems on the height and the leg of the triangle and what is alarming, they do not know to devise a rational part of the line segment length.

Additional interesting constructions

In this section, we would like to show quite a few other simple constructions with a length of a line segments resulting from the use of a finite number of algebraic operations over the field of real numbers. These constructions can be constructed using compasses and a linear. René Descartes (1596-1650) dealt with this issue as well in his book Geometry. (see [1])

Example 1: Let *a* and *b* be the lengths of a line segment. Construct a line segment of the length ab, \sqrt{a} , b/a.

Using line segments of the lengths a, b, the line segment of length 1 and the similarity of triangles, we can easily construct the line segment of the length ab, b/a. If we use, for example, Euclidean theorem on the height, we can construct a line segment of length \sqrt{a} as shown in the following figure.



Figure 4: Constructions the line segment of lenghts *ab*, b/a, \sqrt{a} .

Furthermore, we would like to mention one more interesting theorem from the book [1], which says that the length of the chord of a circle with a radius of length 1 corresponding with the angle α can be determined by the formula: $d = |AB| = \sqrt{2 - 2 \cos \alpha}$.



Figure 5

The proof of this theorem is simple and is based on the use of the cosine theorem within an isosceles triangle *ASB*. This theorem shows that the length of the side of a regular hexagon is equal to the size of the radius of the circle inscribed into this hexagon.

Students learn this particular piece of knowledge in the context of teaching geometry. However, it is unfortunate that they do not learn the general formulation of this theorem, which inter alia implies, that the length of the side of a regular pentagon inscribed in a circle of radius 1 is $\frac{1}{2}\sqrt{10-2\sqrt{5}}$.

Example 2: Let *a* and *b* be the lengths of sides of a rectangle. Construct such a line segment whose length is equal to the k-multiple of lengths *a*, *b*, where k = 1/2, 1/3, 1/4, 1/5.

Only some of our students know how to structurally solve this task by using the similarity of triangles as shown in Fig. 6 for 1/4.



Figure 6: Division of line segment into quarters

It is a great pity that the construction shown in Fig. 7, which was previously massively used in the construction of buildings and the roof trusses, has completely disappeared.

This simple procedure can be proven using the similarity of triangles, which we constructed using this procedure.



Figure 7: Division of line segment into halves, quarters, thirds, fifths etc.

As displayed in Fig. 7, a further us of this procedure results in more multiples, i.e. sixth, eighth, tenth, twelfth, etc. This construction could be included under the theory of fractals, which has been lately given much more attention.

Conclusion

We can only hope that these days, in age of modern technology, the classical approach and its beauty, which gave us great mathematicians and scholars of our past does not completely disappear from today's teaching of geometry. It would be a shame, if the presented constructions so often used in the past and in practice were completely forgotten.

We think it can help do just that, when preparing future teachers we will be given increased attention of hereby and other interesting constructions, for example, during teaching Euclidean geometry in plane.

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