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SUDOKU GAME SOLUTION BASED ON GRAPH THEORY AND SUITABLE FOR SCHOOL-MATHEMATICS

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ABSTRACT. *This article focuses on the logical-mathematical didactic game Sudoku. Analysis of individual fields filling possibilities is mainly based on Graph theory. Ideas, procedures and methods presented in this paper are not demanding and they can be transmitted to secondary school students. In this article the rules of the game and winning strategies analysis derived from Graph theory are mentioned as well as the reasons why this game can be considered a logical game.*

KEY WORDS: *didactic game, Graph theory, winning strategy, logical thinking*

CLASSIFICATION: *D40, A20, E30*

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Introduction

This article is based on the idea that the most natural activity for a student is a game[2] and, therefore, it would be useful for students to use mathematical or didactic games while teaching mathematics[3]. It is a generally known fact that games are often included in teaching for motivational reasons (Vidermanová & Uhrinová, 2011). In this article there are a number of strategies described which can be used when filling the individual fields in Sudoku game. These strategies are primarily based on Graph theory and are so simple that it is possible to use them in teaching this discipline.

Mathematical-logical didactic game Sudoku

The principle of this game is, to some extent, similar to the simple Magic squares (sum of the values in line, column and diagonal is same), which arose as early as about 1000 BC. It is also similar to the Latin Squares (in a square $n \times n$ used n symbols so that in each row and each column each symbol occurs exactly once) from Swiss mathematician Leonard Euler (1707 – 1783). The aim of the game is to fill in the missing numbers 1 – 9 in a predefined, partially pre-filled table which is divided into 9 × 9, i.e. 81 squares. These squares are divided into nine 3 × 3 blocks as figure 1 demonstrates. Filling the table with the numbers must follow these rules:

- Numbers in rows are not repeated
- Numbers in columns are not repeated
- Numbers in 3 × 3 blocks are not repeated

[2] A game is a free action or employment within a clearly defined time and place which is held in freely accepted but, at the same time, unconditionally binding rules. The goal is in itself and it carries with it a sense of tension and joy and, at the same time, awareness of difference from everyday life (Huizinga, 2000).
[3] Martin Gardner defined a mathematical game in Scientific American in a following way: A mathematical game is a multiplayer game whose rules, strategies, and outcomes can be studied and explained by mathematics (Gardner, 1979).

- Order of the numbers when filling is not important

For this game there is no general unambiguous winning strategy. Therefore, the suggestion based on Graph theory how to fill in all the squares will be described. The game should have an unambiguous solution, however, Sudoku without a clear solution are published as well. An example of such a task can be found in fig. 1 where the color-coded numbers are the task. For this task there are four possible solutions that can be obtained by combining possibilities for *A, B, C, D* so that the following is fulfilled:

$$A,B \in \{7;8\} \wedge A \neq B \wedge C,D \in \{5;6\} \wedge C \neq D .$$

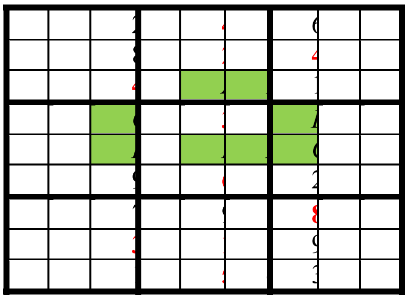

Figure 1: Sudoku without a clear solution

This game can be, to some extent, considered a logical game since while playing it the pupils mainly use the following three logical connections – conjunction, disjunction and consequence. The comment bellow corresponds to 4 × 4 Sudoku in figure 2. Possible considerations could be described in language of logic as follows:

Square *A*: According to the rules this square could be filled with number 3 **or** number 4.

Square *B*: *If* the block rule is ***used together*** with the column rule, ***then*** there is exactly one symbol that can be written in this square.

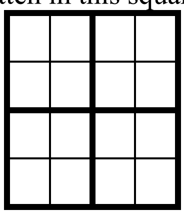

Figure 2: Example of 4 x 4 Sudoku

Sudoku and Graph theory

One of the possible ways to find a winning strategy for Sudoku game is the use of Graph theory. The first option is to use vertex graph coloring.

Vertex graph coloring $G = (V(G), E(G))$ is called the projection $c : V(G) \rightarrow S,\ S \subseteq N$ in which for all nodes u in graph

$\{u,v\} \in E(G) \Rightarrow c(u) \neq c(v)$ is valid. Elements of S set are called colors. Analogously, edge coloring can also be defined as that which will be continuously used. A coloring using at most *k* of different colors is called *k*-coloring. Graph *G* is *k*-colorable if there is *s*-coloring of graph *G* for $s \leq k$.

A Sudoku problem can be solved as a graph coloring problem when we try to color the graph using nine colors assuming that some of these colors are already used. The vertices of the graph will correspond with individual squares and the edges of the graph will link vertices that correspond to squares of one group.

Graf $G = (V(G), E(G))$ corresponding to the classic 9 × 9 Sudoku assignment will thus have 81 vertices that can be described as organized pairs of numbers 1 - 9.

$$V(G) = C \times C \text{ where } C = \{1, 2, \dots, 9\}$$

Set of edges can be described as:

$$E(G) = \left\{ [[x;y];[x';y']]; x = x' \vee y = y' \vee \left(\left\lceil \frac{x}{3} \right\rceil = \left\lceil \frac{x'}{3} \right\rceil \wedge \left\lceil \frac{y}{3} \right\rceil = \left\lceil \frac{y'}{3} \right\rceil \right) \right\},$$

Where $\lceil \ \rceil$ refers to the whole to part.

In this graph each vertex has degree 20, thus the number of edges is: $|\mathrm{H}| = \frac{20.81}{2} = 810$.

Solving the common 9 × 9 task this way is certainly possible but it is too lengthy. The graph coloring issue, therefore, is presented on a simpler version of Sudoku 4 × 4 (figure 3) for which the corresponding graph is constructed (figure 4).

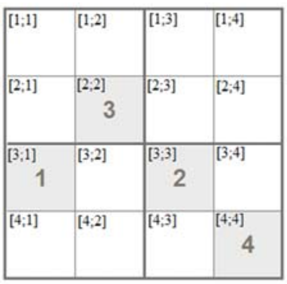

Figure 3: Sample version of 4 × 4 Sudoku

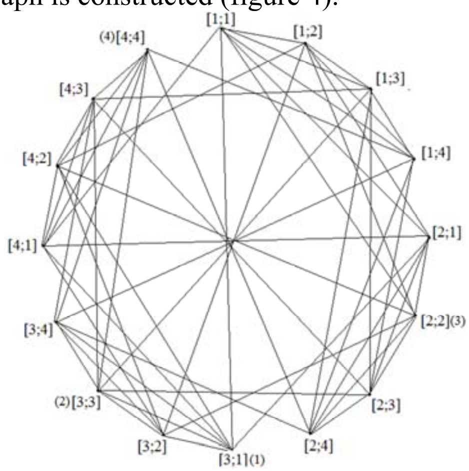

Figure 4: Graph corresponding to 4 × 4 Sudoku task

The above mentioned graph has 16 vertices and 56 edges. The vertices of this graph are numbered the same way as the individual squares in Sudoku assignment. Numerals in square brackets refer to the numbers that were already filled in the Sudoku.

For color labeling the numbers are continuously used depending on how the individual squares are pre-filled and a general heuristic algorithm is used.

General heuristic algorithm for graph coloring

1. In the graph the vertices that are not assigned to any color are found. If there is no color assigned to any vertex, then the algorithm ends.
2. From the set of vertices found on the basis of the first rule, the one that is adjacent to the maximal number of already colored vertices is chosen. When all vertices are colored, the solution is found and the algorithm ended.
3. From the set of vertices chosen with the second rule, the one that is adjacent to the largest number of uncolored vertices is chosen and colored by the color with the lowest value which was not used on its neighbors. If there are more such vertices one of them is selected randomly. In the next step the whole procedure is repeated from the very beginning. When all vertices are colored, the solution is found and the algorithm ended.

In this way, the position [3; 4] in the graph would be assigned color (3) because this vertex neighbors all the other colors. Subgraph[4] [3;1] [3;2] [3;3] and [3;4] is a complete graph and thus the vertex with coordinates [3;2] must be colored with color (4). In this way, the entire graph can be colored.

In the case of a general[5] graph, two problems can appear:

- This procedure cannot be applied because coloring of such a graph by four colors does not exist.
- The coloring exists but it cannot be found with this algorithm.

Such a method of manual processing is quite laborious and unclear because even in the simple assignment it is not visible enough which vertex is linked with all the other colored vertices. For this reason this graph coloring technique was partially changed and named Step after step.

Step after step method

This algorithm of graph coloring describes the ordinary way of thinking while solving Sudoku. The vertex graph coloring is combined with the edge graph coloring. Vertices of the graph are placed in the plane the same way the squares are placed in Sudoku. The edges of the complete subgraphs corresponding to individual groups are not plotted (the edges corresponding to pairs of vertices that cannot be colored with the same color are not plotted). It is, however, assumed that the solver is aware of which vertices are connected by edges. The whole algorithm can be divided into the following steps:

1. The vertex that is already colored is selected and linked by edges of same color with all other vertices of sets in which the vertex is located. These vertices can no longer be colored with the same color. This is repeated for all the vertices for which hints are given[6].

[4] A subgraph of a graph *G* is a graph whose vertex set is a subset of that of *G*, and whose adjacency relation is a subset of that of *G* restricted to this subset. In the other direction, a supergraph of a graph *G* is a graph of which *G* is a subgraph.

[5] In a mathematician's terminology, a graph is a collection of points and lines connecting some (possibly empty) subset of them.

[6] **Hint** is a pre-filled number in game assignment.

2. The vertices where the largest number of colored edges converge are found (it is most likely that there will be only one candidate[7]).
3. If there are vertices among them that can be colored only by one color, then they are colored with it and the procedure continues from the first step (there is no need to draw those edges into the graph that lead to a vertex where there is already another edge of the same color). If there are no such vertices, the procedure continues with the fourth step.
4. From the set of those selected vertices, the one that is adjacent to the largest number of uncolored vertices is chosen and colored to the color with the lowest value that is not used for its neighbors. If there are more such vertices one of them is selected randomly. In the next step the procedure continues from the first step.

Graphic form of Sudoku assignment (figure 3) is shown in figure 5. These edges that correspond to the vertices from the assignment were added (step 1).

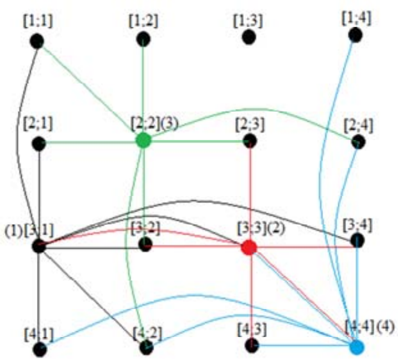

Figure 5: Sudoku graph after step one

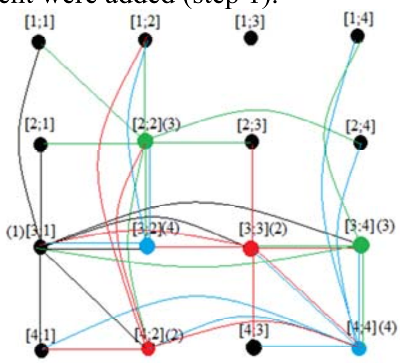

Figure 6: Sudoku graph after step two

In step two (also figure 5) three vertices can be found where lines of three colors converge and it is clear that this vertex must be colored with the last fourth color. Vertex [4;2] must be colored with color (2), vertex [3;4] with color (3) and vertex [3;2] with color (4). In step 3 the given vertices will be colored and relevant edges drawn (figure 6). If, for instance, vertex [4;2] is given color (2) it will not be lined by edge of this color with vertices [3;1] and [3;2]. Similarly, this would proceed with all the other vertices. The procedure will be repeated from step 1.

Based on the two above mentioned methods, it is possible to present issues related to Graph theory in an easy way.

In order to implement this strategy into a computer the matrix representation is used (table 1). In this table the first column lists all vertices of the graph together with the given colors. In each row there is a list of all neighbors of the vertex. This table corresponds to the solution of Sudoku by Step after step method. Similarly to the previous case it is clear that the lines corresponding to vertices [3;2], [3;4] and [4;2] are colored with three different colors and the vertices will, thus, be colored

[7] **Candidate** is a integer from the range 1 to 9 that can be added into an empty square. In this case, the candidate is a number from the range 1 to 4.

with the last, fourth color. The procedure then continues the same as in Step after step method.

List of vertices	Neighboring vertices						
[1;1]	[1;2]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[3;1] (1)	[4;1]
[1;2]	[1;1]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[3;2]	[4;2]
[1;3]	[1;1]	[1;2]	[1;4]	[2;3]	[2;4]	[3;3] (2)	[4;3]
[1;4]	[1;1]	[1;2]	[1;3]	[2;3]	[2;4]	[3;4]	[4;4] (4)
[2;1]	[1;1]	[1;2]	[2;2] (3)	[2;3]	[2;4]	[3;1] (1)	[4;1]
[2;2] (3)	[1;1]	[1;2]	[2;1]	[2;3]	[2;4]	[3;2]	[4;2]
[2;3]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[2;4]	[3;3] (2)	[4;3]
[2;4]	[1;3]	[1;4]	[2;1]	[2;2] (3)	[2;3]	[3;4]	[4;4] (4)
[3;1] (1)	[1;1]	[2;1]	[3;2]	[3;3] (2)	[3;4]	[4;1]	[4;2]
[3;2]	[1;2]	[2;2] (3)	[3;1] (1)	[3;3] (2)	[3;4]	[4;1]	[4;2]
[3;3] (2)	[1;3]	[2;3]	[3;1] (1)	[3;2]	[3;4]	[4;3]	[4;4] (4)
[3;4]	[1;4]	[2;4]	[3;1] (1)	[3;2]	[3;3] (2)	[4;3]	[4;4] (4)
[4;1]	[1;1]	[2;1]	[3;1] (1)	[3;2]	[4;2]	[4;3]	[4;4] (4)
[4;2]	[1;2]	[2;2] (3)	[3;1] (1)	[3;2]	[4;1]	[4;3]	[4;4] (4)
[4;3]	[1;3]	[2;3]	[3;3] (2)	[3;4]	[4;1]	[4;2]	[4;4] (4)
[4;4] (4)	[1;4]	[2;4]	[3;4]	[4;1]	[4;2]	[4;3]	[3;3] (2)

Table 1: Matrix representation of Sudoku assignment

Another way of describing Sudoku solution is the use of independent sets.

Independent set

Set $A \subseteq V(G)$ is called an independent set of graph G if no two vertices of set A are linked by an edge. If it is possible to decompose the set of vertices of an acyclic[8] graph into two independent sets, the graph is called a bipartite[9] graph. If it is possible to decompose it into precisely k of independent sets, it is called a k-partite graph.

Each solution of Sudoku (9 × 9 version) can be described by nine independent subsets of the graph. Each of them has nine vertices labeled with the same number. Finding such a solution that would correspond to the task is, nevertheless, complicated.

Conclusion

This article concentrates on three methods of Sudoku solving based on use of Graph theory. Two of these methods were described in detail. Step after step method was created by the author of this article. Searching Sudoku solutions through Graph theory is considered an appropriate tool aimed to the clarification of the basic theoretical concepts in this field in the classroom. In a similar way, it is also possible to use Nim and base the solution search on the definition of the core of the graph and Sprague-Grundy theorem.

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[8] An acyclic graph is a graph having no graph cycles. Acyclic graphs are bipartie .

[9] A bipartite graph, also called a bigraph, is a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent.

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