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# ON WEAK ISOMETRIES IN ABELIAN DIRECTED GROUPS

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**ABSTRACT.** *In the paper abelian directed groups with the basic intrinsic metric are assumed to be metric spaces and weak isometries, i. e. mappings preserving the basic intrinsic metric, are investigated. The main result is that for each weak isometry* $f$ *in an abelian directed group* $G$ *the relation* $f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$ *is valid for each* $x, y \in G$, *where* $U(x, y)$ *is the set of all upper bounds and* $L(x, y)$ *the set of all lower bounds of the set* $\{x, y\}$ *in* $G$. *This proposition generalizes a result of J. Rachůnek concerning isometries in 2-isolated abelian Riesz groups. Further, the notion of a subgroup symmetry is introduced and it is shown that subgroup symmetries and translations are two basic kinds of weak isometries in 2-isolated abelian directed groups and that each weak isometry in a 2-isolated abelian directed group is a composition of a subgroup symmetry and a translation.*

**KEY WORDS:** *directed group, intrinsic metric, weak isometry*

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## Introduction

Isometries in abelian lattice ordered groups were introduced and studied by K. L. N. Swamy [10], [11]. J. Jakubík [1], [2] considered isometries in non-abelian lattice ordered groups. Isometries in abelian distributive multilattice groups were dealt with by M. Kolibiar and J. Jakubík in [4]. Weak isometries in lattice ordered groups were introduced by J. Jakubík [3]. J. Rachůnek [9] generalized the notion of an isometry to any po-group and investigated isometries in 2-isolated abelian Riesz groups. Weak isometries in some types of po-groups were studied in [5], [6].

## Preliminaries

We review some notions and notations used in the paper.

Let $G$ be a partially ordered group (po-group). The group operation will be written additively. If $A \subseteq G$, then we denote by $U(A)$ and $L(A)$ the set of all upper bounds and the set of all lower bounds of the set $A$ in $G$, respectively. For $A = \{a_1, ..., a_n\}$ we shall write $U(a_1, ..., a_n)$ ($L(a_1, ..., a_n)$) instead of $U(\{(a_1, ..., a_n\})$ ($L(\{(a_1, ..., a_n\})$, respectively). If for $a, b \in G$ there exists the least upper (greatest lower) bound of the set $\{a, b\}$ in $G$, then it will be denoted by $a \vee b$ ($a \wedge b$, respectively).

The set of all subsets of a po-group $G$ will be denoted by *exp* $G$.

A po-group $G$ is called directed if $U(a, b) \neq \emptyset$ and $L(a, b) \neq \emptyset$ for each $a, b \in G$.

A po-group $G$ is called 2-isolated if $2a \geq 0$ implies $a \geq 0$ for each $a \in G$.

A mapping $g$ of a po-group $G$ into $G$ is called an involutory mapping (or an involution) if $g(g(x)) = x$ for each $x \in G$. We will write $g^2(x)$ instead of $g(g(x))$.

The absolute value $|x|$ of an element $x$ of a po-group $G$ is defined by $|x| = U(x, -x)$. If a po-group $G$ is a lattice ordered group, then usually $|x| = x \vee (-x)$ for each $x \in G$.

Recall that if for elements $x$ and $y$ of a po-group $G$ there exists $x \vee y$ in $G$, then there also exist $x \wedge y$, $-x \vee -y$, $-x \wedge -y$ in $G$ and $(x \vee y) + c = (x + c) \vee (y + c)$, $c + (x \vee y) = (c + x) \vee (c + y)$ for each $c \in G$. Moreover, $-(x \vee y) = (-x) \wedge (-y)$. The dual assertions are valid, too.

**Weak isometries in abelian directed groups**

Swamy [10] showed that in any abelian lattice ordered group H the mapping

$$d\colon H \times H \to H \text{ defined by } d(x, y) = |x - y|$$

is an intrinsic metric (or an autometrization) in $H$, i. e. satisfies the formal properties of a distance function:

($M_1$) $d(x, y) \geq 0$ with equality if and only if $x = y$ (positive definiteness),
($M_2$) $d(x, y) = d(y, x)$ (symmetry),
($M_3$) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

J. Rachůnek [9] generalized the notion of an intrinsic metric for any po-group. He defined an intrinsic metric in a po-group $G$ as a mapping $d\colon G \times G \to exp\, G$ satisfying the following conditions for each $x, y, z \in G$:

($\underline{M}_1$) $d(x, y) \subseteq U(0)$ and $d(x, y) = U(0)$ if and only if $x = y$,
($\underline{M}_2$) $d(x, y) = d(y, x)$,
($\underline{M}_3$) $d(x, y) \supseteq d(x, z) + d(z, y)$

and showed that in any 2-isolated abelian Riesz group $H$ the mapping $d\colon H \times H \to exp\, H$ defined by $d(x, y) = |x - y|$ is an intrinsic metric in $H$.

In [7] it was shown that $d(x, y) = |x - y|$ is an intrisic metric in any 2-isolated abelian po-group. This intrinsic metric is called a basic intrinsic metric.

An important notion in the study of structures with a metric is a metric preserving mapping.

If $G$ is a po-group, then a mapping $f\colon G \to G$ is called a weak isometry in $G$ if

$$|f(x) - f(y)| = |x - y| \quad \text{for each } x, y \in G.$$

Hence, a weak isometry in a 2-isolated abelian po-group is a mapping which preserves the basic intrinsic metric $d(x, y) = |x - y|$.

A weak isometry $f$ is called a stable weak isometry if $f(0) = 0$.

A weak isometry $f$ is called an isometry if $f$ is a bijection.

In [5] it was proved that any weak isometry in a directed group is a bijection. Hence the notions of an isometry and of a weak isometry coincide in any directed group.

If $G$ is a po-group and $d \in G$, then a mapping $f\colon G \to G$ such that $f(x) = x + d$ for each $x \in G$ is called a translation in $G$.

This definition of a translation is in close analogy with definition of a translation in the Euclidean plane $R^2$.

The identity mapping can be considered as a special translation with $d = 0$. Every translation in a po-group is an isometry.

If $f$ is a weak isometry in a po-group $G$, then the mapping $g$ defined by $g(x) = f(x) - f(0)$ for each $x \in G$ is a stable weak isometry in $G$. Hence $f(x) = g(x) + f(0)$ for each $x \in G$. If we put $h(x) = x + f(0)$ for each $x \in G$, then $f(x) = h(g(x))$ for each $x \in G$.

Therefore each weak isometry in a po-group is a composition of a stable weak isometry and a translation.

In [9] J. Rachůnek proved that for any isometry $f$ in a 2-isolated abelian Riesz groups $G$ the following condition

$$f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$$

is satisfied for each $x, y \in G$ (Theorem 2.2).

We will extend this proposition to all abelian directed groups.

**Theorem 1.** Let $G$ be an abelian directed group and $g$ a weak isometry in $G$. Let $x, y \in G$, $x \geq y$. Then $g(x) + x \geq g(y) + y$, $-g(x) + x \geq -g(y) + y$.

Proof: Let $x, y \in G$, $x \geq y$. Then from $|x - y| = |g(x) - g(y)|$ we get $U(x - y) = U(x - y, y - x) = U(g(x) - g(y), g(y) - g(x))$. This implies $x - y = (g(x) - g(y)) \vee (g(y) - g(x))$. Thus $x - y \geq g(x) - g(y)$, $x - y \geq g(y) - g(x)$. Hence $-g(x) + x \geq -g(y) + y$, $g(x) + x \geq g(y) + y$.

As a consequence of Theorem 1 we obtain:

**Corollary 1.** If $G$ is an abelian directed group, $g$ a stable weak isometry in $G$ and $x \in G$, $x \geq 0$, then $g(x) + x \geq 0$, $-g(x) + x \geq 0$.

**Theorem 2.** Let $G$ be an abelian directed group and $f$ a weak isometry in $G$. Let $x, y \in G$, $y \leq x$. Then $f([y, x]) = [y - x + f(y),\ x - y + f(y)] \cap [y - x + f(x),\ x - y + f(x)]$.

Proof: Let $x, y \in G$, $y \leq x$. Let $a \in [y, x]$. Thus $y \leq a \leq x$. This yields $f(a) + y \leq f(a) + a \leq f(a) + x$, $-f(a) + y \leq -f(a) + a \leq -f(a) + x$.

From Theorem 1 it follows that $f(y) + y \leq f(a) + a$. Since $f(a) + a \leq f(a) + x$, we have $f(y) + y \leq f(a) + x$. Hence $y - x + f(y) \leq f(a)$.

According to Theorem 1, $-f(y) + y \leq -f(a) + a$. From this and $-f(a) + a \leq -f(a) + x$ we get $-f(y) + y \leq -f(a) + x$. Thus $f(a) \leq x - y + f(y)$.

By Theorem 1, $-f(a) + a \leq -f(x) + x$. Since $-f(a) + y \leq -f(a) + a$, we have $-f(a) + y \leq -f(x) + x$. Then $y - x + f(x) \leq f(a)$.

In view of Theorem 1 we have $f(a) + a \leq f(x) + x$. This and $f(a) + y \leq f(a) + a$ yields $f(a) + y \leq f(x) + x$. Thus $f(a) \leq x - y + f(x)$.

Therefore $f([y, x]) \subseteq [y - x + f(y),\ x - y + f(y)] \cap [y - x + f(x),\ x - y + f(x)]$.

Let $b \in [y - x + f(y),\ x - y + f(y)] \cap [y - x + f(x),\ x - y + f(x)]$. Since each weak isometry in directed group is a bijection, there exists $c \in G$ such that $f(c) = b$. Then $y - x + f(y) \leq f(c) \leq x - y + f(y)$ and hence $f(y) - f(c) \leq x - y$, $f(c) - f(y) \leq x - y$.

Then we obtain $x - y \in U(f(y) - f(c), f(c) - f(y)) = |f(c) - f(y)| = |c - y| = U(c - y, y - c)$. Thus $x - y \geq c - y$. This implies $x \geq c$.

Since $y - x + f(x) \leq f(c) \leq x - y + f(x)$, we have $f(x) - f(c) \leq x - y$, $f(c) - f(x) \leq x - y$. Hence $x - y \in U(f(x) - f(c), f(c) - f(x)) = |f(c) - f(x)| = |c - x| = U(c - x, x - c)$. Thus $x - y \geq x - c$. This yields $c \geq y$. Thus $c \in [x, y]$ and hence $b = f(c) \in f([x, y])$. Therefore $[y - x + f(y),\ x - y + f(y)] \cap [y - x + f(x),\ x - y + f(x)] \subseteq f([y, x])$.

**Theorem 3.** Let $G$ be an abelian directed group and $f$ a weak isometry in $G$. Then the following condition.

(C) $$f(U(L(x, y)) \cap L(U(x, y))) = U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$$

is satisfied for each $x, y \in G$.

Proof: Each weak isometry in $G$ is a composition of a stable weak isometry and a translation. By Theorem 2.7 [9], any translation in a po-group satisfies the condition (C). Thus it suffices to consider that f is a stable weak isometry in $G$.

Let $x, y \in G$, $a \in U(L(x, y)) \cap L(U(x, y))$. Let $u \in L(f(x), f(y))$, $v \in U(f(x), f(y))$. Thus $f(x), f(y) \in [u, v]$.

By Theorem 3 [5], each stable weak isometry in a directed group is an involutory group automorphism. Then in view of Theorem 2 we have $u - v + f(u) \leq f^2(x) = x \leq v - u + f(u)$, $u - v + f(v) \leq f^2(x) = x \leq v - u + f(v)$, $u - v + f(u) \leq f^2(y) = y \leq v - u + f(u)$, $u - v + f(v) \leq f^2(y) = y \leq v - u + f(v)$. Hence $v - u + f(u)$, $v - u + f(v) \in U(x, y)$, $u - v + f(u)$, $u - v + f(v) \in L(x, y)$. From this follows that $u - v + f(u) \leq a \leq v - u + f(u)$, $u - v + f(v) \leq a \leq v - u + f(v)$. Hence $a \in [u - v + f(u), v - u + f(u)] \cap [u - v + f(v), v - u + f(v)]$. Since $a = f^2(a)$, from Theorem 2 it follows that $f(a) \in [u, v]$. Since u was an arbitrary element of $L(x, y)$ and v an arbitrary element of $U(x, y)$, we have $f(a) \in U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$.

Therefore $f(U(L(x, y)) \cap L(U(x, y))) \subseteq U(L(f(x), f(y))) \cap L(U(f(x), f(y)))$.

Then for $x_1 = f(x)$, $y_1 = f(y)$ we get $f(U(L(x_1, y_1)) \cap L(U(x_1, y_1))) \subseteq U(L(f(x_1), f(y_1))) \cap L(U(f(x_1), f(y_1)))$. Since $f$ is an involutory bijection, we have $U(L(x_1, y_1)) \cap L(U(x_1, y_1)) = f^2(U(L(x_1, y_1)) \cap L(U(x_1, y_1))) \subseteq f(U(L(f(x_1), f(y_1))) \cap L(U(f(x_1), f(y_1)))) = f(U(L(f^2(x), f^2(y))) \cap L(U(f^2(x), f^2(y)))) = f(U(L(x, y)) \cap L(U(x, y)))$. Hence $U(L(f(x), f(y))) \cap L(U(f(x), f(y))) \subseteq f(U(L(x, y)) \cap L(U(x, y)))$. This ends the proof.

**Corollary 2.** Let $G$ be an abelian directed group, $f$ a weak isometry in $G$, $x, y \in G$. If there exist $x \wedge y$, $x \vee y$, $f(x) \wedge f(y)$ and $f(x) \vee f(y)$ in $G$, then $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

Proof: If there exist $x \wedge y$, $x \vee y$, $f(x) \wedge f(y)$ and $f(x) \vee f(y)$ in $G$, then $U(L(x, y)) \cap L(U(x, y)) = [x \wedge y, x \vee y]$, $U(L(f(x), f(y))) \cap L(U(f(x), f(y))) = [f(x) \wedge f(y), f(x) \vee f(y)]$. Then from Theorem 3 it follows that $f([x \wedge y, x \vee y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

From Corollary 2 we immediately obtain the following assertions.

**Corollary 3.** Let $G$ be an abelian directed group, $f$ a weak isometry in $G$, $x, y \in G$, $x \leq y$.

(i) If $f(x) \leq f(y)$, then $f([x, y]) = [f(x), f(y)]$.

(ii) If $f(y) \leq f(x)$, then $f([x, y]) = [f(y), f(x)]$.

(iii) If there $f(x) \wedge f(y)$ and $f(x) \vee f(y)$ exist in G, then $f([x, y]) = [f(x) \wedge f(y), f(x) \vee f(y)]$.

Since any Riesz group is a directed group, Theorem 3 generalizes Theorem 2.2 of J. Rachůnek [9].

Elements $x$ and $y$ of a lattice ordered group are called orthogonal if $|x| \wedge |y| = 0$.

J. Rachůnek [12] introduced the following notion of disjointness in po-groups as a generalization of the notion of orthogonality in lattice ordered groups.

Elements $x$ and $y$ of a po-group $H$ are called disjoint (notation $x\delta y$) if there exist $x_1 \in |x|$ and $y_1 \in |y|$ such that $x_1 \wedge y_1 = 0$.

In close analogy with the definition of an axial symmetry in the Euclidean plane $R^2$ we introduce the following notion of a subgroup symmetry.

Let $G$ be a po-group, $A$ a subgroup of $G$ and $d$ the basic intrinsic metric in $G$. A mapping $f\colon G \to G$ is called a symmetry with the respect to subgroup $A$ if the following conditions are satisfied for each $x \in A$, $y \in G \setminus A$:

(i) $f(x) = x$, $f(y) \neq y$,

(ii) $d(f(y), x) = d(y, x)$,
(iii) $(y - f(y))\delta x$.

**Theorem 4.** Let $G$ be an abelian directed group and $f$ a stable weak isometry in $G$. Let $x, y \in G$, $x, y \geq 0$. Then $(-f(x) + x) \wedge (f(y) + y) = 0$.
Proof: Let $z \in G$, $z \geq 0$. Then from $|z - 0| = |f(z) - f(0)|$ it follows that $z = (-f(z)) \vee f(z)$. This implies $(-f(z)) \wedge f(z) = -z$. From this we obtain $(-f(z) + z) \wedge (f(z) + z) = 0$.

Let $x, y \in G, x, y \geq 0$, $v \in U(x, y)$. Then from Theorem 1 it follows that $0 = -f(0) + 0 \leq -f(x) + x \leq -f(v) + v$, $0 = f(0) + 0 \leq f(y) + y \leq f(v) + v$. Since $(-f(v) + v) \wedge (f(v) + v) = 0$, we have $(-f(x) + x) \wedge (f(y) + y) = 0$.

**Theorem 5.** Let $G$ be a 2-isolated abelian directed group and $f$ a stable weak isometry in $G$. Let $A = \{ x \in G;\ f(x) = x\}$, $B = \{x \in G; f(x) = -x\}$. Then $A, B$ are directed convex subgroups of $G$.
Proof. Let $x, y \in A, z \in G, x \leq z \leq y$. Thus $f(x) = x, f(y) = y$. By Corollary 3(i), $f([x, y]) = [x, y]$. Hence $x \leq f(z) \leq y$. Then from $U(y - z) = |y - z| = |f(y) - f(z)| = |y - f(z)| = U(y - f(z))$ it follows that $y - z = y - f(z)$. This implies $f(z) = z$. Hence $A$ is a convex subset of $G$.

Let $a, b \in A$. Since $f$ is a group homomorphism, we have $f(a - b) = f(a) - f(b) = a - b$. Hence $a - b \in A$. Therefore $A$ is a subgroup of $G$.

Further, $G$ is a directed group and hence there exists $u \in G$, such that $u \geq a - b, u \geq 0$. Since $f$ is an involutory group homomorphism, we have $f(u + f(u)) = f(u) + f^2(u) = f(u) + u$. Thus $u + f(u) \in A$. In view of Theorem 1 and Corollary 1 we obtain $f(u) + u \geq f(a - b) + a - b$, $f(u) + u \geq 0$. Then $2(f(u) + u) \geq f(a - b) + a - b = 2(a - b)$. Since group $G$ is 2-isolated, we get $f(u) + u \geq a - b$. Then $f(u) + u + b \in A, f(u) + u + b \geq a, f(u) + u + b \geq b$. Thus $A$ is a directed subgroup of $G$.

Let $x, y \in B, z \in G, x \leq z \leq y$. Since $f(x) = -x, f(y) = -y$, in view of Corollary 2(i) we obtain $f([x, y]) = [-y, -x]$. Hence $-y \leq f(z) \leq -x$. Then from $U(y - z) = |y - z| = |f(y) - f(z)| = |-y - f(z)| = U(f(z) + y)$ we get $y - z = f(z) + y$. This yields $f(z) = -z$. Hence $B$ is a convex subset of $G$.

Let $c, d \in B$. Since $f$ is a group homomorphism, we have $f(c - d) = f(c) - f(d) = -c + d = -(c - d)$. This implies $c - d \in B$. Therefore $B$ is a subgroup of $G$.

Since $G$ is a directed group, there exists $v \in G$, such that $v \geq c - d$, $v \geq 0$. Since $f$ is an involutory group homomorphism, we have $f(v - f(v)) = f(v) - f^2(v)) = f(v) - v = -(v - f(v))$. Thus $v - f(v) \in B$. By Theorem 1 and Corollary 1, $-f(v) + v \geq -f(c - d) + c - \mathrm{d}$, $-f(v) + v \geq 0$. Then $2(-f(v) + v) \geq -f(c - d) + c - d = 2(c - d)$. Since group $G$ is 2-isolated, we have $-f(v) + v \geq c - d$. Then $-f(v) + v + d \in B$, $-f(v) + v + d \geq c, -f(v) + v + d \geq d$. Hence $B$ is a directed subgroup of $G$.

**Theorem 6.** Each stable weak isometry in a 2-isolated abelian directed group $G$ is a subgroup symmetry.
Proof. Let $f$ be a stable weak isometry in $G$. Let $A = \{x \in G; f(x) = x\}$, $B = \{x \in G; f(x) = -x\}$. We will show that $f$ is a symmetry with respect to $A$. Clearly $f(y) \neq y$ for each $y \in G\backslash A$. Let $y \in G\backslash A$, $x \in A$. Then $d(y, x) = d(f(y), f(x)) = d(f(y), x)$. Further, we have $f(y - f(y)) = f(y) - f^2(y) = f(y) - y = -(y - f(y))$. Thus $y - f(y) \in B$. Analogously $f(y) - y \in B$. By Theorem 5, $A$ and $B$ are directed groups. Hence there exist $u \in B$, $v \in A$ such that $u \in U(y - f(y), f(y) - y, 0)$, $v \in U(x, -x, 0)$.

By Theorem 4, $(u - f(u)) \wedge (v + f(v)) = 0$. Thus $(2u) \wedge (2v) = 0$ and hence $u \wedge v = 0$. This yields $(y - f(y))\delta x$. Therefore $f$ is a symmetry with respect to subgroup $A$.

**Corollary 4**. Each weak isometry in a 2-isolated abelian directed group is a composition of a subgroup symmetry and a translation**.**

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