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FROM PROJECTIVE GEOMETRY TO COMPUTER GRAPHICS

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ABSTRACT. *In the lecture we will study the segment of geometry dealing with concept of infinity, the origin of the projective geometry and follow-through the way which leads to geometric modelling of special curves and surfaces in computer graphics.*

KEY WORDS: *point at infinity, projective geometry, curves and surfaces in CAGD*

CLASSIFICATION: *G10, G90*

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Introduction

For study geometric objects for computer graphics and modelling is necessary some knowledge of projective geometry. However the geometry of real objects is Euclidean, the geometry of imaging an object is projective. Hence the study of computer graphics naturally involves both geometries [7]. Projective geometry is useful in two levels: one is such architects use projective geometry when drawing a building as it would appear to an observer. Computer graphic use it for modelling realistic scenes. The second level of use is projective geometry settings for the theory of modelling curves and surfaces, because many of their intrinsic properties are naturally understood in a projective context.

What is projective geometry?

The roots of projective geometry go back to the middle ages. It was in 1425 that the Italian architect Filippo Brunelleschi began to discuss the geometrical theory of perspective, which was consolidated into a treatise a few years later [1]. The concept of perspective occurs besides architecture in painting and astronomy, too.

If we compare the paintings of the Renaissance painters with the painters of the preceding period, we notice a difference in depiction of depth. The figures on Gothic pictures are placed beside one another without any attempt to capture the depth of the space. It was the advent of the renaissance that led artists to study the techniques necessary for realistic rendering. The Renaissance painters wanted to paint the world as they saw it, to paint it from a particular point of view, to paint in perspective, to evoke the illusion of depth [4, 15]. Albrecht Dürer (1471 – 1528) has several works (woodcuts), where he shows a method for create a perspective 2-dimensional map of the 3-dimensional object. One of them is the woodcut "*A man drawing a lute*" (Nürnberg, 1525). This figure shows the scientific approach that Dürer took in order to master perspective: he used wires to record the perceived position of points that were marked on an object [4]. Central projection, illustrated on Dürer's pictures, forms the fundamental idea of projective geometry.

In central projection is a correspondence among points of the object and points of the image, which is established by associating to each point of the object the point of intersection of the image plane with the line containing the object point and the eye. For example, a pair of railroad tracks that disappear off into the distance, an artist adds a vanishing point to the picture. A vanishing point, in general, is that point in a picture at which two parallel lines in the scene appear to meet [17]. Then if we add a point at infinity, or ideal point of mentioned railroad tracks (and similarly for any direction of the object plane, and simultaneously in the image plane) the correspondence between the object and image planes becomes a one-to-one correspondence between the object plane completed with its ideal points and the image plane completed with its ideal points.

The important concept of point at infinity occurred independently to the German astronomer Johann Kepler (1571-1630) and the French architect Girard Desargues (1591-1661). Kepler (in his *Paralipomena in Vitellionem*, 1604) declared that a parabola has two foci, one of which is infinitely distant in both of two opposite directions, and that any point on the curve is joined to this “blind focus” by a line parallel to the axis. Desargues (in his *Brouillon project ...,* 1639) declared that parallel lines have a common end at an infinite distance, and again, “When no point of a line is at a finite distance, the line itself is at an infinite distance” [1]. Then the groundwork was laid to derive projective space from ordinary space by adding a common point at infinity for all lines parallel to each other and adding a common line at infinity for all planes to a given plane. Jean Victor Poncelet (1788-1867) fought in Napoleon’s Russian campaign (1812) until the Russians took him prisoner. As a prisoner at Saratoff on the Volga (1812-1814) he still had the vigour of spirit to implement a great work, he decided to reconstruct the whole science of geometry. The result was his epoch-making “*Traité des propriétés projectives des figures*“, published eight years later, in 1822 [2]. In this work was first made prominent the power of central projection. His leading idea was the study of projective properties, and as a foundation principle he introduced the anharmonic ratio (today known as cross-ratio) [16]. The discovery of the principle of duality was also claimed by Poncelet. This principle of geometric reciprocation has been greatly elaborated and has found its way into modern algebra and elementary geometry [16].

Our list of basic ideas of projective geometry is not exhaustive. We have just mentioned the fundamental ideas, which we need it as a tool for the description of rational curves and surfaces.

Early history of curves and surfaces

The earliest recorded use of curves in a manufacturing environment seems to go back to early AD Roman times, for the purpose of shipbuilding [5]. The vessel’s basic geometry has not changed for a long time. These techniques were perfected by the Venetians from the 13th to the 16th century. No drawings existed to define a ship hull; these became popular in England in the 1600s. The classical “spline”, a wooden beam which is used to draw smooth curves, was probably invented then. The earliest available mention of a “spline" seems to be from 1752 [5].

Another key event originated in aeronautics, where classical drafting methods were combined with computational techniques for the first time.

Some other early influential development for curves and surfaces was the advent of numerical control in the 1950s. In the U.S., General Motors used first CAD (Computer Aided Design) system developed by C. de Boor and W. Gordon. M. Sabin had key role in developing the CAD system for British Aircraft Corporation. He received his PhD from the Hungarian Academy of Sciences in 1977. Sabin developed many algorithms that were later “reinvented” [5].

A new concept

In 1959, the French car company Citroen hired a young mathematician Paul de Faget de Casteljau, who had just finished his PhD. He began to develop a system for design of curves and surfaces with using of Bernstein polynomials. The breakthrough insight was to use control polygons, a technique that was newer used before. De Casteljau's work was kept secret by Citroen for a long time.

During the early 1960s, Pierre Bézier headed the design department at Rénault, the competitor of Citroen, also located in Paris. Bézier's idea was to represent a basic curve as the intersection of two elliptic cylinders placed inside a parallelepiped [6]. Affine transformation of this parallelepiped would result the desired change of the curve (on affine map of the curve). Later, when Bézier used polynomial formulations of the initial concept, the result turned out to be identical to de Casteljau's curves; only the mathematics involved was different [5].

The de Casteljau algorithm for Bézier curves

The de Casteljau algorithm is the most fundamental algorithm in curve and surface modelling, but it is surprisingly simple. It is the beautiful interplay between geometry and algebra. A very intuitive geometric construction leads to a powerful theory [6].

Let us start with the *four tangent theorem* for conic in projective plane [3, 4, 11]. If one tangent is a line at infinity, we get the *three tangent theorem* for parabola [3] in affine plane: Let t_1, t_2, t_3 be three tangents of a parabola in tangent points V_0, V_0^2, V_2, respectively. Let the tangents at V_0 and V_2 intersect in V_1. Let the tangent at V_0^2 intersects the remaining tangents in V_0^1 and V_1^1 (Figure 1).

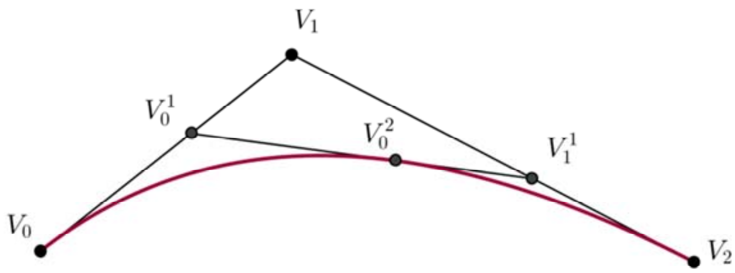

Figure 1

Then the following ratios are equal $(V_0V_1V_0^1) = (V_1V_2V_1^1) = (V_0^1V_1^1V_0^2)$.
It implies that

$$V_0^1 = (1-t)V_0 + tV_1,\ V_1^1 = (1-t)V_1 + tV_2 \text{ and } V_0^2 = (1-t)V_0^1 + tV_1^1,\ t\epsilon\langle 0,1\rangle.$$

Then after calculation we have the point of the parabola given as a barycentric combination of the points V_0, V_1, V_2.

$$V_0^2 = (1-t)^2V_0 + 2(1-t)tV_1 + t^2V_2$$

This is the simplest approach to the essential idea of de Casteljau algorithm. Point V_0^2, the result of the algorithm, is the point of the quadratic Bézier curve given as a linear combination of quadratic Bernstein polynomials. Generalization for degree n gives that a point $X(t)$ on the Bézier curve is given as

$$X(t) = \sum_{i=0}^{n} V_iB_i^n(t),$$

where $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ are Bernstein polynomials of degree n and $t\epsilon\langle 0,1\rangle$. The initial points $V_0, \ldots, V_n$ are the so-called control points of the curve.

Rational Bézier curves

Projective geometry approach allows us to consider some control points at infinity (appear as "control vectors") [4]. Then we get a new type of curve with new possibilities of design it.
Another approach is to define Bézier curve $X(t) = \sum_{i=0}^{n} \mathbf{V_i}\, B_i^n(t)$ in projective space with control points $\boldsymbol{V_i} = [w_i V_i, w_i]$ in P^3, and to map it into the embedded affine space. In the analytic expression of the curve it yields that

$$X(t) = \frac{\sum_{i=0}^{n} w_i V_i B_i^n(t)}{\sum_{i=0}^{n} w_i B_i^n(t)}, \qquad t\epsilon\langle 0,1\rangle,$$

where V_i are control points in affine space and w_i are the so-called weights of the curve. This type of curve is known as a rational Bézier curve, because contains the ratio of two polynomials. Rational Bézier curves may be evaluated equally, using the concept of cross ratios, the fundamental invariant of projective geometry. Application of Menelaus` theorem leads to the same result [10, 11].
Rational curves have several advantages over polynomial Bézier curves. A degree two polynomial Bézier curve can only represent a parabola. Exact representation of circles and all conic sections requires rational degree two Bézier curves. The shape of the curve can be influenced not only with the shape of control polygon, but also with appropriate weights. A perspective projection of a Bézier curve is a rational Bézier curve.

Duality

A basic concept of projective geometry, the duality concept is widely used in curve modelling [13, 18]. Dual counterpart of plane Bézier curve defined by control points is a Bézier curve defined by control lines. The curve is thus given as the envelope of its tangents. In 3D space the dual curve is defined by control planes. The importance of this dual concept is in the theory of developable surface. The key observation is, that while the planes $\chi(t)$ are osculating planes of a curve, they themselves, being a one parameter family of planes, envelope a surface [4]. Such surfaces are called developable.

Triangular Bézier patches

In surface modelling, the finite piece of surface is called a patch. Two basic types are tensor product patches and triangular Bézier patches (Bézier triangles). When de Casteljau invented Bézier curves in 1959, he realized the need for the extension of the curve ideas for surfaces. The first surface type that he considered was what we now call Bézier triangles. This historical first of triangular patches is reflected by the mathematical statement that they are more natural generalization of Bézier curves than are tensor product patches [6]. Thus Bézier triangles can be perceived as a generalization of Bézier curves (for triangular domain instead of unit interval used for curves). Let a parameter $U = (u,v,w)$ be an element of triangular domain, where $0 \le u, v, w \le 1$ are barycentric coordinates. Expression of the Bézier triangle is then very similar to expression of the Bézier curve:

$$X(U) = \sum_{i+j+k=n} V_{ijk} B_{ijk}^n(U),$$

where V_{ijk} are control points and $B_{ijk}^n(U)$ are trivariate Bernstein polynomials

$$B_{ijk}^{n}(U) = B_{ijk}^{n}(u,v,w) = \frac{n!}{i!\,j!\,k!} u^{i} v^{j} w^{k}$$

($i, j, k \in \{0, 1, \dots, n\}$ and all subscripts sum to n).
Consider now a projective Bézier triangle; the previous polynomial Bézier triangle defined in projective space. Following the familiar theme of generating rational curve, we define a rational Bézier triangle as the projection of polynomial Bézier triangle to affine space.

$$X(U) = \frac{\sum_{i+j+k=n} w_{ijk} V_{ijk} B_{ijk}^{n}}{\sum_{i+j+k=n} w_{ijk} B_{ijk}^{n}},$$

where w_{ijk} are weights associated with the control points V_{ijk}, describes the rational Bézier triangle.

Quadrics

While the quadratic polynomial Bézier triangle represents a part of paraboloid (if it fulfils some extra condition [6, 8, 9,14]), their rational counterpart allows us to represent a part of quadric (hyperboloid, ellipsoid or especially sphere), because every quadric surface may be defined as a projective image of a paraboloid.
We get the following characterization for quadratic rational Bézier triangle lying on quadric surface [6, 8, 14]: A rational quadratic Bézier triangle is a part of quadric surface if and only if extensions of all tree their boundary curves meet in a common point of the quadric and have coplanar tangents there.
Using the previous condition and the so-called Patchwork Theorem [9] for degenerated 4-sided patches, it is possible to cover a sphere with combination of these patches [14]. (Figure 2 shows the set of triangular patches fulfilling the previous condition on the sphere, drawn with program Maple.)

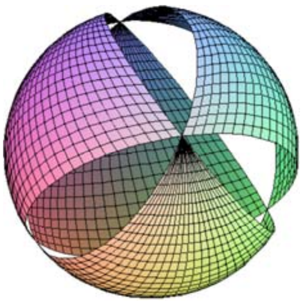

Figure 2

Summary

Projective geometry is a natural setting for many types of curves and surfaces used in computer aided geometric design (CAGD). The aim of this paper was to show the way from the beginnings of the projective geometry to its using in CAGD nowadays. Naturally, the overview is not complete, very important curves (e.g. NURBS, B-splines) and methods (e.g. stereographic projection [7, 18], WRD-construction [12]) and many others were not mentioned.

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